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A METHOD FOR DEFLECTION ANALYSIS OF THIN LOW-ASPECT-RATIO WINGS

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SUMMARY

A method is presented for obtaining influence coefficients for thin low-aspect-ratio wings of built-up construction. Chordwise deflections are assumed to be parabolic (or linear) and the principle of minimum potential energy is used in conjunction with difference equivalents to obtain an appropriate set of equilibrium equations. Symmetric and antisymmetric load-support conditions are considered. A simple method is given for taking account of attachment to a flexible fuselage. The computations involved are for the most part organized into matrix form so as to be suitable for high-speed computing machines. The matrices needed at the beginning of the computing process can be set up directly from the data of the wing design.

INTRODUCTION

The simple beam theory based on the concept of an elastic axis with uncoupled bending and twisting distortions has long been successfully applied to high-aspect-ratio unswept wings. However, the assumptions upon which the simple beam theory is based invalidate its application to low-aspect-ratio wings. In the case of a low-aspect-ratio wing, bending and torsion are not uncoupled; among other things, there appears the additional complication of chordwise bending. Thus, the development of an adequate load-deflection analysis for use in predicting the aeroelastic behavior of an airplane with low-aspect-ratio wings has become an important structural problem.

Several approaches to the solution of this problem have appeared in the recent literature. In a method introduced by Levy (ref. 1), the wing is idealized into a network of crisscrossing beams to represent ribs and spars with four-sided torsion boxes attached at their four corners to the intersections of the beams to represent the shear-carrying capacity of the skin. (See fig. 1.) An effective width of the skin is allotted to the beams to account for its direct-stress-carrying capacity. A stiffness matrix (or inverse influence-coefficient matrix) is found for the composite structure by properly combining the stiffness matrices of the various components. The influence-coefficient matrix of the composite structure is obtained by inverting the stiffness matrix of the composite structure. The method is meant to be applicable to thin wings with thin cover sheets.

In the "wide beam" theory introduced by Schuerch (ref. 2), the wing is idealized into a bundle of alternating simple beams and torsion tubes running in the spanwise direction. (See fig. 2.) The beams represent the spars and axial-load-carrying capacity of the skin. The torsion tubes represent the shear-carrying capacity of the skin. The bundle is tied together by ribs which are assumed rigid. The individual beams and torsion tubes are assumed to behave according to simple beam theory. The equations of moment and torque equilibrium for the idealized structure are two simultaneous differential equations (both of the fourth order) for the deflection and rotation of the cross sections. These equations are not uncoupled as in simple beam theory, and several new section properties appear. This method, presumably, is meant to be applied to thin wings with thin cover sheets. In reference 3 Schuerch and Freelin have presented another method which is somewhat like that of Levy's.

In recent papers, Williams (refs. 4 and 5) has outlined a method for thick-skinned thin wings with closely spaced spars and ribs (the effectiveness of which is included in the skin). In this method, the partial-differential equations of plate theory are assumed to be applicable. The differential equations are replaced by difference equations in which the unknowns are the deflections at a large number of lattice points spread over the wing. The method is very much like a relaxation method except the "relaxing" is done all at once and for any load distribution by means of a matrix inversion.

Other methods of a quite different nature exist for analyzing low-aspect-ratio wings, namely analog methods. These range anywhere from testing a model of the wing to building an electrical analog of an idealized wing structure as in the method of MacNeal and Benscoter (ref. 6).

The method to be described in the present paper is an analytical one and is essentially an extension to built-up wings of a method developed for thin solid wings by Reissner and Stein in reference 7 and by Stein, Anderson, and Hedgepeth in reference 8. The chief difference between the theories of Levy, Schuerch, and Williams and the present theory is that they simplify the problem by idealizing the structure whereas the approach of the present paper is to simplify the problem by idealizing the deformations. The application of the method is therefore not limited to any specific range of skin thicknesses.

The plate-like nature of the thin low-aspect-ratio wing suggests the adoption of some of the assumptions of plate theory. Accordingly, in this paper, the displacements of the material points of the wing are assumed to be expressible in terms of the vertical displacements of a neutral surface. (See fig. 3.) The material points on a normal to the undeformed neutral surface are assumed to remain on that normal during deformation of the wing. Furthermore, the deflections of the neutral surface are assumed to be either linear or parabolic in the chordwise direction (both theories are treated in this report), but are otherwise unrestricted. An appropriate set of equilibrium equations is obtained by using the foregoing specification of displacements in conjunction with the principle of minimum potential energy. By approximating the energy integral by a finite sum in which all derivatives in the integrand are expressed in difference form, the minimization process leads to equilibrium equations in matrix form.

Specific directions for setting up the required matrices from the "raw" data (for example, from drawings of the wing) are given. Two load-support conditions are considered fundamental, namely, symmetric and antisymmetric. In each of these two cases, a set of influence coefficients is derived for a wing that is assumed to be supported in a particular way specifically chosen to facilitate routine application of the method. (See fig. 4.) A procedure is then given whereby these influence coefficients may be modified to obtain those appropriate to many other kinds of support including pin-jointed mounting on a flexible fuselage.

SYMBOLS

a_k stiffness coefficient for cover plates, $\int_{c_1(y)}^{c_2(y)} Dx^k dx$

$c_1(y)$, $c_2(y)$ functions defining plan form (see fig. 5)

D local flexural stiffness of wing given by equation (5)

E modulus of elasticity of material

I moment of inertia of spar or rib

i, j, k, m, n, r, s integers

l semispan

N number assigned to station at tip of wing, $l = N\epsilon$

$P_{n,m}$ load at mth reference point on chord at nth station

p	transverse load per unit area
t	thickness of a cover plate
t_u, t_l	thicknesses of upper and lower cover plates, respectively
$W_{n,m}$	deflection at mth reference point on chord at nth station
w	lateral deflection of wing (see fig. 3)
x, y, z	coordinates defined in figure 3
$x_s(y)$	function defining location of sth spar or stringer (see fig. 5)
y_r	spanwise location of rth rib or stiffener
z_u, z_l	distance from neutral surface to upper and lower cover plates, respectively
β	spar or stringer stiffness coefficient
γ_r	rth rib or stiffener stiffness coefficient
ϵ	strain in stringer; distance between equally spaced stations
ϵ_x, ϵ_y, γ_{xy}	components of normal and shearing strain in cover plates
ζ, θ	coefficients of rigid body translation and pitch, respectively
λ	angle of sweep of spar or stringer (see fig. 5)
μ	Poisson's ratio
$\xi_{n,m}$	distance from y-axis to mth reference point on chord at nth station (see fig. 7)
$\varphi_k(y)$	deflection coefficients (see eq. (1))
Π	energy

METHOD OF ANALYSIS

The structure considered in this paper is a thin wing of low aspect ratio built up of cover sheet, spars, ribs, and stiffeners. The carry-through structure is considered to be part of the wing. The wing is supposed to be internally stiffened so that transverse shear deflections can be neglected. It will be assumed that there is a neutral surface which does not stretch (or at least that its stretching is negligible). Then, since transverse shears are neglected, normals to this neutral surface will remain normal during bending of the wing. Under these conditions, the displacements of all material points in the wing are expressible in terms of the lateral deflections of the neutral surface, assumed to be given by the equation (see fig. 3)

$$w = \varphi_0(y) + x\varphi_1(y) + x^2\varphi_2(y) \tag{1}$$

The potential-energy function which is to be minimized is the difference between the strain energy and twice the work of the external loads. But before the minimization process is carried out the potential-energy function is expressed in discrete form by writing all derivatives as differences and all integrations as numerical integrations (by the use of the trapezoidal rule or suitable variations thereof). The unknowns in the expression then become the values of φ_0, φ_1, and φ_2 at a number of stations along the span. After properly accounting for geometric boundary conditions, minimization of the potential-energy function with respect to each of the unknowns leads to a system of simultaneous algebraic equations which can be written in matrix form. The unknowns in this system of equations are the φ_k values at the station points; terms on the right-hand sides of these equations are derived from the loading. The system is solved by a matrix inversion.

Strain- and Potential-Energy Expressions

The first step in the procedure is the calculation of the strain energy of the wing in terms of the deflection of the neutral surface. The energies of the various components of the wing will be considered separately.

Covers.- The strain energy of stetching of one cover plate (say the upper) is given by

$$\frac{1}{2}\int_0^l \int_{c_1}^{c_2} \frac{Et_u}{1 - \mu^2}\left[\left(\epsilon_x + \epsilon_y\right)^2 + 2(1 - \mu)\left(\frac{1}{4}\gamma_{xy}{}^2 - \epsilon_x\epsilon_y\right)\right] dx\ dy \qquad (2)$$

The coordinate system chosen in the plan form of the wing is shown in figure 5 together with the definitions of c_1, c_2, and l. The thickness of the upper cover plate t_u may be a function of x and y. According to the fundamental assumptions, the middle-surface strains in the cover are given in terms of the deflection w of the neutral surface by

$$\epsilon_x = -z_u\frac{\partial^2 w}{\partial x^2} \qquad \epsilon_y = -z_u\frac{\partial^2 w}{\partial y^2} \qquad \gamma_{xy} = -2z_u\frac{\partial^2 w}{\partial x\ \partial y} \qquad (3)$$

where $z_u \equiv z_u(x,y)$ is the distance from the neutral surface of the wing to the middle surface of the upper cover plate (z_u is assumed to be a slowly varying function of x and y). In terms of w, the strain energy of stretching of both cover plates is

$$\Pi_c = \frac{1}{2}\int_0^l \int_{c_1}^{c_2} D\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1 - \mu)\left[\left(\frac{\partial^2 w}{\partial x\ \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\} dx\ dy \qquad (4)$$

where

$$D(x,y) = \frac{E}{1 - \mu^2}\left(t_u z_u{}^2 + t_l z_l{}^2\right) \qquad (5)$$

in which the subscripts u and l refer to the upper and lower covers, respectively. The strain energy of bending of the cover plates can be neglected if the thickness of the cover plates is small in comparison with the thickness of the wing. This simplification has been assumed here.

When the expression for w given by equation (1) is introduced into equation (4), the integration with respect to x may be carried out; the result is

$$\Pi_c = \frac{1}{2}\int_0^l \Big\{a_0\left(\varphi_0''\right)^2 + 2a_1\varphi_0''\varphi_1'' + a_2\left[\left(\varphi_1''\right)^2 + 2\varphi_0''\varphi_2''\right] +$$

$$2a_3\varphi_1''\varphi_2'' + a_4\left(\varphi_2''\right)^2 + 4a_0\varphi_2^2 + 4\mu\left(a_0\varphi_0'' + a_1\varphi_1'' + a_2\varphi_2''\right)\varphi_2 +$$

$$2(1 - \mu)\left[a_0\left(\varphi_1'\right)^2 + 4a_1\varphi_1'\varphi_2' + 4a_2\left(\varphi_2'\right)^2\right]\Big\} dy \qquad (6)$$

where the primes indicate differentiation with respect to y and

$$a_k(y) = \int_{c_1}^{c_2} Dx^k dx \qquad (7)$$

<u>Spars and stringers</u>.- The strain energy of a spar (or stringer) running from y_0 to y_1 at an angle λ to the y-axis is

$$\frac{1}{2}\int_{\rho(y_0)}^{\rho(y_1)} EI(\rho)\left(\frac{d^2\bar{w}}{d\rho^2}\right)^2 d\rho = \frac{1}{2}\int_{y_0}^{y_1} \beta\left(\frac{d^2\bar{w}}{dy^2}\right)^2 dy \qquad (8)$$

where ρ is distance along the spar,

$$\beta = EI(y)\cos^3\lambda \qquad (9)$$

and $\bar{w}(y) = w\left[x(y),y\right]$ where $x = x(y)$ is the equation of the line along which the spar lies (see fig. 5). The moment of inertia I is calculated for sections normal to the lengthwise direction of the spar about an axis lying in the neutral surface of the wing. In terms of φ_0, φ_1, and φ_2, the strain-energy expression for a spar (or stringer) is

$$\Pi_s = \frac{1}{2}\int_{y_0}^{y_1} \beta_s\left[\varphi_0'' + \left(x_s\varphi_1\right)'' + \left(x_s^2\varphi_2\right)''\right]^2 dy \qquad (10)$$

where the subscript s is used to identify the spar or stringer.

Ribs and chordwise stiffeners.- The strain energy of a rib (or chordwise stiffener) located at $y = y_r$ is

$$\Pi_r = \frac{1}{2}\int_{c_1(y_r)}^{c_2(y_r)} EI_r\left(\frac{\partial^2 w}{\partial x^2}\right)^2_{y=y_r} dx \tag{11}$$

where $I_r(x)$ is the moment of inertia of the rib about the neutral surface. In terms of φ_2, the expression for Π_r is

$$\Pi_r = \frac{1}{2}\gamma_r\varphi_2{}^2(y_r) \tag{12}$$

where

$$\gamma_r = 4\int_{c_1(y_r)}^{c_2(y_r)} EI_r\ dx \tag{13}$$

Loads.- The potential-energy function of the transverse loads of intensity p is

$$\Pi_p = -\int_0^l \int_{c_1}^{c_2} pw\ dx\ dy \tag{14}$$

A more explicit form for Π_p when the loads are concentrated is given later.

Finite-Difference Forms

The potential-energy function which is to be minimized to obtain appropriate equilibrium equations is

$$\Pi = \Pi_c + \sum_s \Pi_s + \sum_r \Pi_r + \Pi_p \tag{15}$$

An application of the calculus of variations at this point would lead to differential equations of equilibrium which could then be put in difference form; however, it is easier to proceed to these directly. The minimization process leads directly to a set of simultaneous linear algebraic equations for the φ_k values at discrete points along the span when the derivatives occurring in the expression for Π are replaced by finite differences and the integrations are replaced by finite sums. In this paper, the stations along the span of the wing are equally spaced and numbered from $n = 0$ at the center line to $n = N$ at the tip. There are also stations corresponding to $n = -1$ and $n = N + 1$. (See fig. 6.) The following difference approximations for first and second derivatives are used:

$$\left.\begin{aligned} \left(\frac{df}{dy}\right)_{n+\frac{1}{2}} &\approx \frac{f_{n+1} - f_n}{\epsilon} \\ \left(\frac{d^2f}{dy^2}\right)_n &\approx \frac{f_{n+1} - 2f_n + f_{n-1}}{\epsilon^2} \end{aligned}\right\} \tag{16}$$

where n indicates the station and ϵ is the distance between stations. Integrals are approximated by finite sums according to the trapezoidal rule or, if necessary, by some appropriate modification thereof. Thus, the discrete form of the contribution to Π_c (see eq. (6)) due to a typical term involving a second derivative is

$$\begin{aligned} \frac{1}{2}\int_0^l a_0\left(\varphi_0''\right)^2 dy = \frac{1}{2\epsilon^3}\Big[&\tfrac{1}{2}a_{0,0}\left(\varphi_{0,-1} - 2\varphi_{0,0} + \varphi_{0,1}\right)^2 + \\ &a_{0,1}\left(\varphi_{0,0} - 2\varphi_{0,1} + \varphi_{0,2}\right)^2 + \cdot \cdot \cdot + \\ &a_{0,N-1}\left(\varphi_{0,N-2} - 2\varphi_{0,N-1} + \varphi_{0,N}\right)^2 + \\ &\tfrac{1}{2}a_{0,N}\left(\varphi_{0,N-1} - 2\varphi_{0,N} + \varphi_{0,N+1}\right)^2\Big] \end{aligned} \tag{17}$$

or due to a typical term involving a first derivative is

$$\frac{1}{2}\int_0^l 2(1 - \mu)a_0(\varphi_1')^2 dy = \frac{1 - \mu}{\epsilon}\left[a_{0,\frac{1}{2}}(\varphi_{1,1} - \varphi_{1,0})^2 + \cdot \cdot \cdot + a_{0,N-\frac{1}{2}}(\varphi_{1,N} - \varphi_{1,N-1})^2\right] \tag{18}$$

The discrete form of the other terms entering into the expression for Π_c may be written in a similar way; they are omitted here for simplicity. These equations are valid if the a_k values are continuous functions of y. However, discontinuities due to cutouts, reinforcement, and so forth may occur, in which case some provision for modifying the trapezoidal rule to improve its accuracy when applied to discontinuous functions should be made. This correction can be made by replacing the values of a_k at the station nearest the discontinuity by modified a_k values according to a simple rule which will be given later. For a typical spar which begins, for example, at station n_0 and ends at station n_1, the energy may be written in the following way. First let

$$\chi_{s,n} = \varphi_{0,n} + x_{s,n}\varphi_{1,n} + x_{s,n}{}^2\varphi_{2,n} \tag{19}$$

$$\psi_{s,n} = \chi_{s,n-1} - 2\chi_{s,n} + \chi_{s,n+1} \tag{20}$$

Then, from equation (10),

$$\Pi_s = \frac{1}{2\epsilon^3}\left(\frac{1}{2}\beta_{s,n_0}\psi_{s,n_0}{}^2 + \beta_{s,n_0+1}\psi_{s,n_0+1}{}^2 + \cdot \cdot \cdot + \beta_{s,n_1-1}\psi_{s,n_1-1}{}^2 + \frac{1}{2}\beta_{s,n_1}\psi_{s,n_1}{}^2\right) \tag{21}$$

which may be written

$$\Pi_s = \frac{1}{2\epsilon^3}\left(\bar{\beta}_{s,0}\psi_{s,0}^{\ 2} + \ . \ . \ . + \bar{\beta}_{s,N}\psi_{s,N}^{\ 2}\right) \tag{22}$$

where the definition of $\bar{\beta}_s$ in terms of β_s is obvious by a comparison of equations (21) and (22). In case the spar does not end exactly at station n_1 (or n_0), the value of $\bar{\beta}_s$ at station n_1 (or n_0) must be redefined to take this into account; the exact definition is given later. In any case, formula (22) will still hold if $\bar{\beta}_{s,n}$ is properly defined.

For a rib at $y_r = (n_0 + d_r)\epsilon$, the energy Π_r is

$$\Pi_r = \frac{1}{2}\gamma_r\left[\left(1 - d_r\right)\varphi_{2,n_0} + d_r\varphi_{2,n_0+1}\right]^2 \tag{23}$$

where $0 \leqq d_r \leqq 1$. Here linear interpolation has been used to obtain a value of φ_2 between stations.

Boundary Conditions

The energy expressions in the last section involve φ_k values at stations -1 and N + 1; the disposition of these terms depends on the boundary conditions. The support system should be chosen so that the boundary conditions take a simple form and so that the generality of the result is not restricted; that is, the influence coefficients appropriate to the chosen support system must be such that influence coefficients appropriate to any other support system can be derived from them (as is not the case for cantilever support, for example). For deriving a set of influence coefficients for symmetric loading, the wing is considered to be clamped at the origin. (See fig. 4(a).) The conditions on φ_k are

$$\varphi_0 = \varphi_1 = \varphi_0{'} = \varphi_1{'} = \varphi_2{'} = 0 \qquad \text{(at } y = 0\text{)} \tag{24}$$

or in difference form

$$\varphi_{0,0} = \varphi_{1,0} = 0 \qquad \varphi_{0,-1} = \varphi_{0,1} \qquad \varphi_{1,-1} = \varphi_{1,1} \qquad \varphi_{2,-1} = \varphi_{2,1} \tag{25}$$

In the antisymmetric case the wing is considered to be simply supported at the points $(x=0, y=\pm\epsilon)$ and constrained (reactionlessly) so that the center line does not deflect. (See fig. 4(b).) In difference form

$$\varphi_{0,0} = \varphi_{1,0} = \varphi_{2,0} = \varphi_{0,-1} = \varphi_{0,1} = 0 \qquad \varphi_{1,-1} = -\varphi_{1,1} \qquad \varphi_{2,-1} = -\varphi_{2,1} \tag{26}$$

At the tip there are no geometrical contraints and hence no restrictions on the values of $\varphi_{k,N}$ or $\varphi_{k,N+1}$ to be enforced. The equations

$$\frac{\partial \Pi}{\partial \varphi_{k,N+1}} = 0 \tag{27}$$

could be used to eliminate the three unknowns $\varphi_{k,N+1}$ from the energy expression, but it is much simpler to retain them as unknowns.

Matrix Form of Equilibrium Equations

The final form of the equilibrium equations obtained by minimizing the potential-energy function Π subject to the boundary conditions will be given in this section. The derivations of the results are mostly a matter of routine but lengthy algebra, and they have been omitted from the paper except for an example derivation given in appendix B.

The equations

$$\frac{\partial \Pi}{\partial \varphi_{k,n}} = 0 \tag{28}$$

lead directly to the matrix equation

$$\left[A\right]\left[\varphi\right] = \epsilon^3\left[p\right] \tag{29}$$

or in expanded form

$$\begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{bmatrix} = \epsilon^3 \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} \tag{30}$$

The definitions of the matrices $\left[A_{ij}\right]$ are given as follows (in which primes indicate the transpose of a matrix):

$$\left.\begin{aligned}
\left[A_{00}\right] &= \left[D_0\right]\left(\left[a_0\right] + \sum_s \left[\bar{\beta}_s\right]\right)\left[D_0\right]' \\
\left[A_{01}\right] &= \left[D_0\right]\left(\left[a_1\right]\left[D_1\right]' + \sum_s \left[\bar{\beta}_s\right]\left[X_s\right]'\right) \\
\left[A_{02}\right] &= \left[D_0\right]\left(\left[a_2\right]\left[D_2\right]' + \sum_s \left[\bar{\beta}_s\right]\left[X_s^2\right]' + 2\mu\epsilon^2\left[\bar{a}_0\right]\right) \\
\left[A_{11}\right] &= \left[D_1\right]\left[a_2\right]\left[D_1\right]' + \sum_s \left[X_s\right]\left[\bar{\beta}_s\right]\left[X_s\right]' + 2(1 - \mu)\epsilon^2\left[D_3\right]\left[a_0^*\right]\left[D_3\right]' \\
\left[A_{12}\right] &= \left[D_1\right]\left[a_3\right]\left[D_2\right]' + \sum_s \left[X_s\right]\left[\bar{\beta}_s\right]\left[X_s^2\right]' + 4(1 - \mu)\epsilon^2\left[D_3\right]\left[a_1^*\right]\left[D_4\right]' + 2\mu\epsilon^2\left[D_1\right]\left[\bar{a}_1\right] \\
\left[A_{22}\right] &= \left[D_2\right]\left[a_4\right]\left[D_2\right]' + \sum_s \left[X_s^2\right]\left[\bar{\beta}_s\right]\left[X_s^2\right]' + 8(1 - \mu)\epsilon^2\left[D_4\right]\left[a_2^*\right]\left[D_4\right]' + \\
&\qquad 2\mu\epsilon^2\left\{\left[D_2\right]\left[\bar{a}_2\right] + \left(\left[D_2\right]\left[\bar{a}_2\right]\right)'\right\} + 4\epsilon^4\left[\hat{a}_0\right] + \epsilon^3\sum_r\left[\Gamma_r\right] \\
\left[A_{10}\right] &= \left[A_{01}\right]' \qquad \left[A_{20}\right] = \left[A_{02}\right]' \qquad \left[A_{21}\right] = \left[A_{12}\right]'
\end{aligned}\right\} \tag{31}$$

The definitions of the matrices entering into the right-hand sides of equations (31) and the $[\varphi]$ and $[p]$ matrices in equation (30) depend on the boundary conditions. They are as follows (where the number of rows and columns, in that order, are indicated below the equation number):

For the symmetric case:

$$[D_0] = [D_1] = \begin{bmatrix} 2 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & & \cdots & & & \\ & & & & & 1 & -2 & 1 \\ & & & & & & 1 & -2 \\ & & & & & & & 1 \end{bmatrix} \qquad \begin{matrix} (32) \\ (N+1)\times(N+1) \end{matrix}$$

$$[D_2] = \begin{bmatrix} -2 & 1 & & & & & & \\ 2 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & & \cdots & & & \\ & & & & & 1 & -2 & 1 \\ & & & & & & 1 & -2 \\ & & & & & & & 1 \end{bmatrix} \qquad \begin{matrix} (33) \\ (N+2)\times(N+1) \end{matrix}$$

$$[D_3] = \begin{bmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & & \cdots & & \\ & & & & 1 & -1 \\ & & & & & 1 \\ & & & & & 0 \end{bmatrix} \qquad \begin{matrix} (34) \\ (N+1)\times N \end{matrix}$$

$$\left[D_4\right] = \begin{bmatrix} -1 & & & & & \\ 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & & \cdots & & \\ & & & & 1 & -1 \\ & & & & & 1 \\ & & & & & 0 \end{bmatrix} \qquad \begin{matrix}(35)\\ (N+2)\times N\end{matrix}$$

$$\left[a_k\right] = \begin{bmatrix} \frac{1}{2}a_{k,0} & & & & & \\ & a_{k,1} & & & & \\ & & a_{k,2} & & & \\ & & & \cdots & & \\ & & & & a_{k,N-1} & \\ & & & & & \frac{1}{2}a_{k,N} \end{bmatrix} \qquad \begin{matrix}(36)\\ (N+1)\times(N+1)\end{matrix}$$

$$\left[\bar{a}_k\right] = \begin{bmatrix} \frac{1}{2}a_{k,0} & & & & & & \\ & a_{k,1} & & & & & \\ & & a_{k,2} & & & & \\ & & & \cdots & & & \\ & & & & a_{k,N-1} & & \\ & & & & & \frac{1}{2}a_{k,N} & 0 \end{bmatrix} \qquad \begin{matrix}(37)\\ (N+1)\times(N+2)\end{matrix}$$

$$\left[a_k^*\right] = \begin{bmatrix} a_{k,\frac{1}{2}} & & & \\ & a_{k,\frac{3}{2}} & & \\ & & \cdots & \\ & & & a_{k,N-\frac{1}{2}} \end{bmatrix} \qquad \begin{matrix}(38)\\ N\times N\end{matrix}$$

$$\left[\hat{a}_0\right] = \begin{bmatrix} \frac{1}{2}a_{0,0} & & & & & & \\ & a_{0,1} & & & & & \\ & & a_{0,2} & & & & \\ & & & \cdots & & & \\ & & & & a_{0,N-1} & & \\ & & & & & \frac{1}{2}a_{0,N} & \\ & & & & & & 0 \end{bmatrix} \qquad \begin{matrix}(39)\\ (N+2)\times(N+2)\end{matrix}$$

$$\left[\bar{\beta}_s\right] = \begin{bmatrix} \bar{\beta}_{s,0} & & & \\ & \bar{\beta}_{s,1} & & \\ & & \cdots & \\ & & & \bar{\beta}_{s,N} \end{bmatrix} \qquad \begin{matrix}(40)\\ (N+1)\times(N+1)\end{matrix}$$

$$\left[X_s\right] = \begin{bmatrix} x_{s,-1}+x_{s,1} & -2x_{s,1} & x_{s,1} & & & & & \\ & x_{s,2} & -2x_{s,2} & x_{s,2} & & & & \\ & & x_{s,3} & -2x_{s,3} & x_{s,3} & & & \\ & & & & \cdots & & & \\ & & & & & x_{s,N-1} & -2x_{s,N-1} & x_{s,N-1} \\ & & & & & & x_{s,N} & -2x_{s,N} \\ & & & & & & & x_{s,N+1} \end{bmatrix}_{(N+1)\times(N+1)} \tag{41}$$

$$\left[X_s^2\right] = \begin{bmatrix} & -2x_{s,0}^2 & x_{s,0}^2 & & & & & \\ x_{s,-1}^2+x_{s,1}^2 & -2x_{s,1}^2 & x_{s,1}^2 & & & & & \\ & x_{s,2}^2 & -2x_{s,2}^2 & x_{s,2}^2 & & & & \\ & & x_{s,3}^2 & -2x_{s,3}^2 & x_{s,3}^2 & & & \\ & & & & \cdots & & & \\ & & & & & x_{s,N-1}^2 & -2x_{s,N-1}^2 & x_{s,N-1}^2 \\ & & & & & & x_{s,N}^2 & -2x_{s,N}^2 \\ & & & & & & & x_{s,N+1}^2 \end{bmatrix}_{(N+2)\times(N+1)} \tag{42}$$

When the rth rib falls between stations, $\left[\Gamma_r\right]$ has the following form:

$$\left[\Gamma_r\right] = \begin{bmatrix} 0 & & & & & & \\ & \cdots & & & & & \\ & & 0 & & & & \\ & & & \gamma_r\left(1-d_r^2\right) & \gamma_r d_r\left(1-d_r\right) & & \\ & & & \gamma_r d_r\left(1-d_r\right) & \gamma_r d_r^2 & & \\ & & & & & 0 & & \\ & & & & & & \cdots & \\ & & & & & & & 0 \end{bmatrix}_{(N+2)\times(N+2)} \tag{43}$$

Rules for proper positioning of the elements in the $\left[\Gamma_r\right]$ matrix are given in the section entitled "Mechanics of Application." The expanded form of the $[\varphi]$ matrices occurring in equation (30) is

$$\left[\varphi_0\right] = \begin{bmatrix} \varphi_{0,1} \\ \varphi_{0,2} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{0,N+1} \end{bmatrix} \qquad \begin{matrix} (44a) \\ (N+1)\times 1 \end{matrix}$$

$$\left[\varphi_1\right] = \begin{bmatrix} \varphi_{1,1} \\ \varphi_{1,2} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{1,N+1} \end{bmatrix} \qquad \begin{matrix} (44b) \\ (N+1)\times 1 \end{matrix}$$

$$\left[\varphi_2\right] = \begin{bmatrix} \varphi_{2,0} \\ \varphi_{2,1} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{2,N+1} \end{bmatrix} \qquad \begin{matrix} (44c) \\ (N+2)\times 1 \end{matrix}$$

Also

$$\left[p_0\right] = \begin{bmatrix} p_{0,1} \\ p_{0,2} \\ \cdot \\ \cdot \\ \cdot \\ p_{0,N} \\ 0 \end{bmatrix} \qquad \begin{matrix}(45a)\\(N+1)\times 1\end{matrix}$$

$$\left[p_1\right] = \begin{bmatrix} p_{1,1} \\ p_{1,2} \\ \cdot \\ \cdot \\ \cdot \\ p_{1,N} \\ 0 \end{bmatrix} \qquad \begin{matrix}(45b)\\(N+1)\times 1\end{matrix}$$

$$\left[p_2\right] = \begin{bmatrix} p_{2,0} \\ p_{2,1} \\ \cdot \\ \cdot \\ \cdot \\ p_{2,N} \\ 0 \end{bmatrix} \qquad \begin{matrix}(45c)\\(N+2)\times 1\end{matrix}$$

where

$$p_{k,n} = -\frac{\partial \Pi_p}{\partial \varphi_{k,n}} \tag{46}$$

For the antisymmetric case:

$$\left[D_0\right] = \begin{bmatrix} 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & & \cdots & & & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \\ & & & & & & 1 \end{bmatrix} \qquad \begin{matrix}(47)\\ N \times N\end{matrix}$$

$$\left[D_1\right] = \left[D_2\right] = \begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & & \cdots & & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \\ & & & & & 1 \end{bmatrix} \qquad \begin{matrix}(48)\\ (N+1) \times N\end{matrix}$$

$$\left[D_3\right] = \left[D_4\right] = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \cdots & & \\ & & & 1 & -1 \\ & & & & 1 \\ & & & & 0 \end{bmatrix} \qquad \begin{matrix}(49)\\ (N+1) \times N\end{matrix}$$

$$\left[a_k\right] = \begin{bmatrix} a_{k,1} & & & & \\ & a_{k,2} & & & \\ & & \cdot\ \cdot\ \cdot & & \\ & & & a_{k,N-1} & \\ & & & & \frac{1}{2}a_{k,N} \end{bmatrix} \qquad \begin{matrix}(50)\\ N\times N\end{matrix}$$

$$\left[\bar{a}_k\right] = \begin{bmatrix} a_{k,1} & & & & & \\ & a_{k,2} & & & & \\ & & \cdot\ \cdot\ \cdot & & & \\ & & & a_{k,N-1} & & \\ & & & & \frac{1}{2}a_{k,N} & 0 \end{bmatrix} \qquad \begin{matrix}(51)\\ N\times(N+1)\end{matrix}$$

$\left[a_k{}^*\right]$ same as symmetric case

$$\left[\hat{a}_0\right] = \begin{bmatrix} a_{0,1} & & & & & \\ & a_{0,2} & & & & \\ & & \cdot\ \cdot\ \cdot & & & \\ & & & a_{0,N-1} & & \\ & & & & \frac{1}{2}a_{0,N} & \\ & & & & & 0 \end{bmatrix} \qquad \begin{matrix}(52)\\ (N+1)\times(N+1)\end{matrix}$$

$$\left[\bar{\beta}_s\right] = \begin{bmatrix} \bar{\beta}_{s,1} & & & \\ & \bar{\beta}_{s,2} & & \\ & & \cdots & \\ & & & \bar{\beta}_{s,N} \end{bmatrix} \qquad \begin{matrix}(53)\\ N\times N\end{matrix}$$

$$\left[x_s\right] = \begin{bmatrix} -2x_{s,1} & x_{s,1} & & & & \\ x_{s,2} & -2x_{s,2} & x_{s,2} & & & \\ & x_{s,3} & -2x_{s,3} & x_{s,3} & & \\ & & & \cdots & & \\ & & & x_{s,N-1} & -2x_{s,N-1} & x_{s,N-1} \\ & & & & x_{s,N} & -2x_{s,N} \\ & & & & & x_{s,N+1} \end{bmatrix} \qquad \begin{matrix}(54)\\ (N+1)\times N\end{matrix}$$

$$\left[x_s^2\right] = \begin{bmatrix} -2x_{s,1}^2 & x_{s,1}^2 & & & & \\ x_{s,2}^2 & -2x_{s,2}^2 & x_{s,2}^2 & & & \\ & x_{s,3}^2 & -2x_{s,3}^2 & x_{s,3}^2 & & \\ & & & \cdots & & \\ & & & x_{s,N-1}^2 & -2x_{s,N-1}^2 & x_{s,N-1}^2 \\ & & & & x_{s,N}^2 & -2x_{s,N}^2 \\ & & & & & x_{s,N+1}^2 \end{bmatrix} \qquad \begin{matrix}(55)\\ (N+1)\times N\end{matrix}$$

The matrix $\left[\Gamma_r\right]$ has a form similar to that of the symmetric case except that it is of order $(N+1)\times(N+1)$. The proper placement of the elements in $\left[\Gamma_r\right]$ will be given in the section entitled "Mechanics of Application." The expanded form of the $[\varphi]$ and $[p]$ matrices occurring in equation (30) for the antisymmetric case is

$$\left[\varphi_0\right] = \begin{bmatrix} \varphi_{0,2} \\ \varphi_{0,3} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{0,N+1} \end{bmatrix} \qquad \begin{matrix} (56a) \\ N\times 1 \end{matrix}$$

$$\left[\varphi_1\right] = \begin{bmatrix} \varphi_{1,1} \\ \varphi_{1,2} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{1,N+1} \end{bmatrix} \qquad \begin{matrix} (56b) \\ (N+1)\times 1 \end{matrix}$$

$$\left[\varphi_2\right] = \begin{bmatrix} \varphi_{2,1} \\ \varphi_{2,2} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{2,N+1} \end{bmatrix} \qquad \begin{matrix} (56c) \\ (N+1)\times 1 \end{matrix}$$

$$\begin{bmatrix} p_0 \end{bmatrix} = \begin{bmatrix} p_{0,2} \\ p_{0,3} \\ \cdot \\ \cdot \\ \cdot \\ p_{0,N} \\ 0 \end{bmatrix} \qquad \begin{matrix} (57a) \\ N\times 1 \end{matrix}$$

$$\begin{bmatrix} p_1 \end{bmatrix} = \begin{bmatrix} p_{1,1} \\ p_{1,2} \\ \cdot \\ \cdot \\ \cdot \\ p_{1,N} \\ 0 \end{bmatrix} \qquad \begin{matrix} (57b) \\ (N+1)\times 1 \end{matrix}$$

$$\begin{bmatrix} p_2 \end{bmatrix} = \begin{bmatrix} p_{2,1} \\ p_{2,2} \\ \cdot \\ \cdot \\ \cdot \\ p_{2,N} \\ 0 \end{bmatrix} \qquad \begin{matrix} (57c) \\ (N+1)\times 1 \end{matrix}$$

Inversion of $[A]$ Matrix

The next step in the derivation is the solution of the system of equation (30). This may be done either by inverting $[A]$ at once or alternatively by first eliminating $[\varphi_0]$ (as in ref. 8) and then inverting a matrix of roughly two-thirds the order of $[A]$. The most economical choice depends on the number of stations and the computing facilities available.

The result of eliminating $[\varphi_0]$ is

$$\left[\begin{array}{c|c} A_{11} - A_{10}A_{00}^{-1}A_{01} & A_{12} - A_{10}A_{00}^{-1}A_{02} \\ \hline A_{21} - A_{20}A_{00}^{-1}A_{01} & A_{22} - A_{20}A_{00}^{-1}A_{02} \end{array}\right] \left[\begin{array}{c} \varphi_1 \\ \hline \varphi_2 \end{array}\right] = \epsilon^3 \left[\begin{array}{c} p_1 - A_{10}A_{00}^{-1}p_0 \\ \hline p_2 - A_{20}A_{00}^{-1}p_0 \end{array}\right] \tag{58}$$

or say

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \epsilon^3 \begin{bmatrix} C_1 & I & 0 \\ C_2 & 0 & I \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} \tag{59}$$

where $[I]$ is the identity matrix. There would hardly be any advantage in eliminating $[\varphi_0]$ if setting up the $[B_{ij}]$ matrices for computation were as complicated a task as it looks. However, the terms which involve products of $[A_{ij}]$ matrices can be considerably simplified and lead to the following results:

$$\left.\begin{aligned}
\left[B_{11}\right] &= \left[A_{11}\right] - \left(\left[D_1\right]\left[a_1\right] + \sum_s \left[X_s\right]\left[\bar{\beta}_s\right]\right)\left(\left[a_0\right] + \sum_s \left[\bar{\beta}_s\right]\right)^{-1}\left(\left[D_1\right]\left[a_1\right] + \sum_s \left[X_s\right]\left[\bar{\beta}_s\right]\right)' \\
\left[B_{12}\right] &= \left[A_{12}\right] - \left(\left[D_1\right]\left[a_1\right] + \sum_s \left[X_s\right]\left[\bar{\beta}_s\right]\right)\left(\left[a_0\right] + \sum_s \left[\bar{\beta}_s\right]\right)^{-1}\left(\left[D_2\right]\left[a_2\right] + \sum_s \left[X_s^2\right]\left[\bar{\beta}_s\right] + 2\mu\epsilon^2\left[\bar{a}_0\right]'\right)' \\
\left[B_{22}\right] &= \left[A_{22}\right] - \left(\left[D_2\right]\left[a_2\right] + \sum_s \left[X_s^2\right]\left[\bar{\beta}_s\right] + 2\mu\epsilon^2\left[\bar{a}_0\right]'\right)\left(\left[a_0\right] + \sum_s \left[\bar{\beta}_s\right]\right)^{-1}\left(\left[D_2\right]\left[a_2\right] + \sum_s \left[X_s^2\right]\left[\bar{\beta}_s\right] + 2\mu\epsilon^2\left[\bar{a}_0\right]'\right)' \\
\left[B_{21}\right] &= \left[B_{12}\right]'
\end{aligned}\right\} \qquad (60)$$

Note that there are essentially only three different matrices within parentheses and that one is the inverse of a diagonal matrix. Also

$$\left.\begin{aligned}
\left[C_1\right] &= -\left(\left[D_1\right]\left[a_1\right] + \sum_s \left[X_s\right]\left[\bar{\beta}_s\right]\right)\left(\left[a_0\right] + \sum_s \left[\bar{\beta}_s\right]\right)^{-1}\left[D_0\right]^{-1} \\
\left[C_2\right] &= -\left(\left[D_2\right]\left[a_2\right] + \sum_s \left[X_s^2\right]\left[\bar{\beta}_s\right] + 2\mu\epsilon^2\left[\bar{a}_0\right]'\right)\left(\left[a_0\right] + \sum_s \left[\bar{\beta}_s\right]\right)^{-1}\left[D_0\right]^{-1}
\end{aligned}\right\} \qquad (61)$$

where in the symmetric case

$$\left[D_0\right]^{-1} = \begin{bmatrix}
\frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & \cdots & & & \frac{N+1}{2} \\
 & 1 & 2 & 3 & 4 & \cdots & & & N \\
 & & 1 & 2 & 3 & \cdots & & & N-1 \\
 & & & & & \cdots & & & \\
 & & & & & & 1 & 2 & 3 \\
 & & & & & & & 1 & 2 \\
 & & & & & & & & 1
\end{bmatrix} \qquad (62)$$

and in the antisymmetric case

$$\left[D_0\right]^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdot\cdot\cdot & & & N \\ & 1 & 2 & 3 & \cdot\cdot\cdot & & & N-1 \\ & & & & \cdot\cdot\cdot & & & \\ & & & & & 1 & 2 & 3 \\ & & & & & & 1 & 2 \\ & & & & & & & 1 \end{bmatrix} \tag{63}$$

In order to complete the solution for $[\varphi]$ in terms of $[p]$, the matrix which must be inverted is

$$\left[B\right] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{64}$$

The solution may then be written

$$\begin{bmatrix} \varphi_0 \\ \hline \varphi_1 \\ \varphi_2 \end{bmatrix} = \epsilon^3 \left(\left[\begin{array}{c|c} \left(C'B^{-1}\right)C & C'B^{-1} \\ \hline \left(C'B^{-1}\right)' & B^{-1} \end{array}\right] + \left[\begin{array}{c|cc} A_{00}{}^{-1} & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \right) \begin{bmatrix} p_0 \\ \hline p_1 \\ p_2 \end{bmatrix} \tag{65}$$

or

$$\left[\varphi\right] = \epsilon^3 \left[A\right]^{-1} \left[p\right] \tag{66}$$

where

$$\left[C\right] = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{67}$$

and

$$\left[A_{00}\right]^{-1} = \left(\left[D_0\right]^{-1}\right)'\left(\left[a_0\right] + \sum_s \left[\bar{\beta}_s\right]\right)^{-1}\left[D_0\right]^{-1} \tag{68}$$

In certain cases, the preceding operations will have to be modified because the matrices which are to be inverted turn out to be singular. In particular, this may happen when the wing has a pointed tip. The reason the matrices are singular is that at some stations the three unknown sets of φ_k values are linearly dependent; in other words, the deflection is overdetermined at that station. The details of the remedy for this situation are given in the section "Mechanics of Application."

After the $\left[A\right]^{-1}$ matrix is obtained, it is modified slightly. According to the derivation there are rows and columns of $\left[A\right]^{-1}$ corresponding to $\varphi_{k,N+1}$ and $p_{k,N+1}$. These three rows and columns are to be deleted. Also, in the symmetric case, it will prove convenient to insert two rows of zeros corresponding to $\varphi_{0,0}$ and $\varphi_{1,0}$ and two columns of zeros corresponding to $p_{0,0}$ and $p_{1,0}$. In the antisymmetric case, insert a row of zeros corresponding to $\varphi_{0,1}$ and a column of zeros corresponding to $p_{0,1}$. The meaning of $\left[\varphi\right]$ and $\left[p\right]$ is altered but the notation will be kept the same in what follows. Also, the factor ϵ^3 (see eq. (30)) is included in the modified $\left[A\right]^{-1}$ matrix. The result is a symmetric matrix which will be called $\left[g\right]$. Equation (66) becomes now

$$\left[\varphi\right] = \left[g\right]\left[p\right] \tag{69}$$

Equation (69) gives generalized deflections in terms of generalized loads and essentially expresses the solution to the load-deflection problem. For some applications (finding modes and frequencies, for instance), it is convenient to proceed from this point keeping the φ_k as generalized deflections. Only the problem of finding influence coefficients will be worked out in detail.

Influence Coefficients

A set of reference points on the plan form of the wing must be chosen with respect to which the matrix of influence coefficients will be given. The reference points on the plan form of the wing are located in terms of the set of numbers $\xi_{n,m}$ which give the distance from the y-axis to the mth reference point on the chord at the nth spanwise station. (See fig. 7.) A quantity referring to a particular reference point will be labeled with the subscripts n,m. There may be any number (α_n) of reference points along the chord at station n but the deflections at only three of the reference points are independent in the parabolic theory (two in linear theory).

The relation of $\left[p\right]$ to the concentrated loads $P_{n,m}$ at the reference points can be obtained through the definition of $p_{k,n}$ in terms of Π_p; that is,

$$p_{k,n} = -\frac{\partial \Pi_p}{\partial \varphi_{k,n}} \tag{70}$$

In terms of $P_{n,m}$ and $W_{n,m}$ (the deflection at reference point (n,m)), the expression for Π_p is

$$\Pi_p = -\sum_{n,m} W_{n,m}P_{n,m} \tag{71}$$

or

$$\Pi_p = -\sum_{n,m} \left(\varphi_{0,n} + \xi_{n,m}\varphi_{1,n} + \xi_{n,m}{}^2\varphi_{2,n}\right)P_{n,m} \tag{72}$$

Then from equation (70),

$$p_{k,n} = \sum_{m} \left(\xi_{n,m}\right)^k P_{n,m} \tag{73}$$

which may be written in matrix form as

$$\left[p\right] = \left[H\right]\left[P\right] \tag{74}$$

or in expanded form, for the symmetric case,

$$
\begin{bmatrix} p_{0,0} \\ p_{0,1} \\ \vdots \\ p_{0,N} \\ \hline p_{1,0} \\ p_{1,1} \\ \vdots \\ p_{1,N} \\ \hline p_{2,0} \\ p_{2,1} \\ \vdots \\ p_{2,N} \end{bmatrix}
=
\left[\begin{array}{cccc:cccc:c:cccc}
1 & 1 & \cdots & 1 & & & & & & & & & \\
 & & & & 1 & 1 & \cdots & 1 & & & & & \\
 & & & & & & & & \cdots & & & & \\
 & & & & & & & & & 1 & 1 & \cdots & 1 \\
\hdashline
\xi_{0,1} & \xi_{0,2} & \cdots & \xi_{0,\alpha_0} & & & & & & & & & \\
 & & & & \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,\alpha_1} & & & & & \\
 & & & & & & & & \cdots & & & & \\
 & & & & & & & & & \xi_{N,1} & \xi_{N,2} & \cdots & \xi_{N,\alpha_N} \\
\hdashline
\xi_{0,1}^2 & \xi_{0,2}^2 & \cdots & \xi_{0,\alpha_0}^2 & & & & & & & & & \\
 & & & & \xi_{1,1}^2 & \xi_{1,2}^2 & \cdots & \xi_{1,\alpha_1}^2 & & & & & \\
 & & & & & & & & \cdots & & & & \\
 & & & & & & & & & \xi_{N,1}^2 & \xi_{N,2}^2 & \cdots & \xi_{N,\alpha_N}^2
\end{array}\right]
\begin{bmatrix} P_{0,1} \\ P_{0,2} \\ \vdots \\ P_{0,\alpha_0} \\ \hline P_{1,1} \\ P_{1,2} \\ \vdots \\ P_{1,\alpha_1} \\ \hline \vdots \\ \hline P_{N,1} \\ P_{N,2} \\ \vdots \\ P_{N,\alpha_N} \end{bmatrix}
\tag{75}
$$

In the antisymmetric case, $[H]$ is given by

$$
[H] =
\left[\begin{array}{cccc:cccc:c:cccc}
1 & 1 & \cdots & 1 & & & & & & & & & \\
 & & & & 1 & 1 & \cdots & 1 & & & & & \\
 & & & & & & & & \cdots & & & & \\
 & & & & & & & & & 1 & 1 & \cdots & 1 \\
\hdashline
\xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,\alpha_1} & & & & & & & & & \\
 & & & & \xi_{2,1} & \xi_{2,2} & \cdots & \xi_{2,\alpha_2} & & & & & \\
 & & & & & & & & \cdots & & & & \\
 & & & & & & & & & \xi_{N,1} & \xi_{N,2} & \cdots & \xi_{N,\alpha_N} \\
\hdashline
\xi_{1,1}^2 & \xi_{1,2}^2 & \cdots & \xi_{1,\alpha_1}^2 & & & & & & & & & \\
 & & & & \xi_{2,1}^2 & \xi_{2,2}^2 & \cdots & \xi_{2,\alpha_2}^2 & & & & & \\
 & & & & & & & & \cdots & & & & \\
 & & & & & & & & & \xi_{N,1}^2 & \xi_{N,2}^2 & \cdots & \xi_{N,\alpha_N}^2
\end{array}\right]
\tag{76}
$$

From the definition of w in terms of φ_k (eq. (1)), the following matrix relation may be deduced:

$$\left[W\right] = \left[H\right]'\left[\varphi\right] \tag{77}$$

where $\left[W\right]$ is a column matrix whose elements $W_{n,m}$ are arranged in the same order as the elements $P_{n,m}$ in $\left[P\right]$. (See eq. (75).) From equations (69), (74), and (77), it is evident that

$$\left[W\right] = \left[G\right]\left[P\right] \tag{78}$$

where

$$\left[G\right] = \left[H\right]'\left[g\right]\left[H\right] \tag{79}$$

The matrix $\left[G\right]$ gives the desired set of influence coefficients.

As defined in this paper, the influence coefficients given by equation (79) are the deflections at symmetrically placed reference points (with respect to the center line) due to symmetrically placed unit loads. There is one instance in which the definition may be a little ambiguous and that is when the load is on the center line in the symmetric case. In this case the deflections at all reference points are to be interpreted as due to a double load on the center line since now the two unit loads, symmetrically placed, have moved into coincidence. The obvious alternate definition would not result in a symmetric $\left[G\right]$ matrix.

Equations of the Linear Theory

The equations of the linear theory may be obtained from those of the parabolic theory by setting $\varphi_{2,n} = 0$ and omitting the equations $\frac{\partial \Pi}{\partial \varphi_{2,n}} = 0$ at the outset. The steps in the subsequent derivation are identical. In this case the matrix which has to be inverted is either

$$\left[A\right] = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \tag{80a}$$

or

$$[B] = [B_{11}] \tag{80b}$$

In the latter alternative $\left(\text{that is, when } [B] \text{ is inverted}\right)$, $[A]^{-1}$ can be obtained from the equation

$$[A]^{-1} = \left[\begin{array}{c|c} C'B^{-1}C & C'B^{-1} \\ \hline \left(C'B^{-1}\right)' & B^{-1} \end{array}\right] + \left[\begin{array}{c|c} A_{00}{}^{-1} & 0 \\ \hline 0 & 0 \end{array}\right] \tag{81}$$

where now

$$[C] = [C_1] \tag{82}$$

The same modification of $[A]^{-1}$ as is described in the paragraph preceding equation (69) is again made to obtain $[g]$. As before

$$[G] = [H]'[g][H] \tag{83}$$

where $[H]$ is the same as in parabolic theory except the last N rows are omitted.

Adaptation to Other Loading or Support Conditions

A knowledge of the load-deflection characteristics of the wing apart from the fuselage is often insufficient for accurate aeroelastic analysis of the airplane. However, the effect of attaching the wing (which includes the carry-through structure) to the fuselage at a number of simple supports (no moments transmitted) can be accounted for by a fairly simple modification of the influence coefficients obtained by using the standardized supports. The details of the analysis are given in appendix A.

In the analysis of the present paper, the carry-through structure has been assumed to be part of the wing. The assumption is valid if the carry-through structure is not very different from the wing structure; for example, this would be the case if one cover sheet and some stringers were cut out but the spars were continued on through. If the carry-through structure is appreciably different from the wing structure, then a modified analysis is called for. The analysis of the present paper could probably be adapted to the case in which each half-wing is attached to a flexible structure characterized by a set of influence coefficients which involve rotations and couples as well as deflections and forces at the points of attachment of the half-wings.

For some applications it is necessary to know the deflections due to a unit couple applied at some point along the trailing edge (as caused by an aileron, for example). If the trailing edge is parallel to the y-axis and the point of application of the couple falls on a station, then the solution to the problem is simple. A reference point is introduced off the trailing edge at the reflection of some reference point on the chord. Equal and opposite loads at these two reference points produce the same effect as a couple applied at the trailing edge within the framework of the parabolic (or linear) theory. The deflection due to a load at the reference point off the wing is found in exactly the same way as though it were on the wing. In case the trailing edge is swept or the couple is applied between stations, the correct procedure for finding the deflections due to the couple can be found by recourse to the energy method $\left(\text{find } \Pi_p\right)$.

DISCUSSION

Approximations Involved in the Theory

The approximations involved in the present theory are, for the most part, consequences of restrictions placed on the displacements at the outset. Some of these approximations, such as the neglect of transverse shear deflections, are quite common and need little discussion but some others, more or less peculiar to the present theory, require some words of justification and some warning as to the limitations they place on the theory.

The restriction to parabolic (or linear) chordwise deflections has already been used successfully in the case of a solid plate (ref. 8). In the present case, the ribs near the root are not expected to bend appreciably because of the restraint offered by the fuselage. For a wing of low aspect ratio, nearly the whole wing is within the region of influence of the root restraint. Also, since the ribs have zero (or, at most, small) bending moment at either end, the chordwise bending moments are much less than the spanwise bending moments so that, if the stiffnesses in the two directions are of the same order, the chordwise curvature should be much less than the spanwise curvature. Thus, provided the ribs are not too light, it seems justifiable to assume that the chordwise deflections may be well approximated by a parabola, or even by a straight line.

If the wing is perfectly symmetrical, top and bottom, so that the middle plane may be taken as the neutral surface, then the stretching of the "neutral" surface is certainly extremely small (zero within the framework of small-deflection theory) when the wing is bent under lateral loads. However, if there are cutouts on the top or bottom of the wing or various other asymmetries, then the situation is not nearly so clean-cut. The difficulty is that there does not necessarily exist a surface which remains unstretched (even in small-deflection theory) when the wing is subjected to a lateral load. As an exaggerated example, suppose there are many spanwise stiffeners on the top of the wing and many chordwise stiffeners on the bottom. In such a case, it would seem reasonable to specify the u and v displacements with respect to two surfaces, one on which u is zero and the other on which v is zero, rather than with respect to a single neutral surface on which both u and v are zero. That such a case would arise in practice is unlikely, but at least it shows the need for making some reasonable assumptions. The only way really to avoid the necessity of making engineering judgments with respect to the choice of a neutral surface would be to go to some more exact theory in which more complicated expressions for specifying displacements are assumed. Barring that, the locus of the principal axis of inertia of the chordwise cross sections seems to be a reasonable choice. The resulting influence coefficients probably will be in error in the neighborhood of a cutout or other discontinuity, but the overall results should be accurate.

The method of analysis of the present paper has been motivated by the resemblance between the low-aspect-ratio wing and the plate inasmuch as they are both thin and flat. However, since the carry-through structure may not even remotely resemble a plate, the validity of the method must be examined on that point. The criterion is whether the assumed form of the displacements can accurately describe the true shape of the deformed structure. If, for example, the carry-through structure is composed of three beams which are the continuation of three wing spars, then the deflections of these three beams can be described exactly in terms of the three functions φ_0, φ_1, and φ_2 of the parabolic theory. In this case, there is no question that the analysis is adequate as far as the carry-through member is concerned. However, the chordwise location of the continuous spars could conceivably be such that the deflected shapes of the inboard chords of the wing could not be approximated very well by a parabola even when the loading on the wing is reasonably well distributed. In addition, the discontinuities in the structure cause discontinuities in the higher derivatives of φ_k and this may cause an appreciable error in the difference equivalents of the lower derivatives used in the analysis.

Refinements Over Beam Theory

Some of the ways in which the present theory falls short of exactness were pointed out in the last section. There are, in favor of the present theory, a number of refinements over beam theory which are either manifest or implicit in the equations. The additional generality of the more refined theory allows for restraint against warping of the root cross section and for restraint against anticlastic curvature at the root. In a long beam these effects are felt only in the region near the root, but in a low-aspect-ratio wing where the root chord may be about as large as the semispan these effects are appreciable over the whole wing. A glance at the form of the equations shows that bending (φ_0), torsion (φ_1), and chordwise bending (φ_2) are inextricably coupled together; however, in simple beam theory, the equations are uncoupled by virtue of the assumption of an elastic axis.

Comparison With Other Theories

In either Levy's or Schuerch's theory the direct-stress-carrying capacity and the shear-carrying capacity of the sheet are separated. The direct-stress-carrying capacity of the sheet is lumped in with the spars and ribs. This is a common assumption in the treatment of semimonocoque structures with thin skin. However, when the cover plates get thicker, and thus form a proportionately greater part of the total wing material, a more refined treatment of them is appropriate. Probably the most important effect neglected in the thin-sheet approximation is the coupling between spanwise and chordwise stretching of the sheet - in other words, the Poisson's ratio effect which produces anticlastic curvature. In the present theory, a nearly exact expression for the strain energy of the cover sheets is used. The accuracy of the result depends on how well the assumed displacements approximate the true ones, but does not depend directly on the thinness of the cover sheets. So, at least in the parabolic theory, the Poisson's ratio effect is taken into account. Williams' method is the antithesis of Levy's in the sense that primary consideration is given to the covers rather than the spars and ribs. However, Williams' method is not well adapted to the analysis of a wing with a few heavy spars and ribs or with thin skin. At the present time the ranges of applicability of the several theories have not been well defined owing partly to a lack of experimental evidence and partly to the newness of the problem.

Some Possible Improvements and Extensions of the Theory

The effect of transverse shear deflections in the ribs and spars is possibly the most serious omission in the present theory as it now stands.

For some configurations the effect of shear deflection in the ribs would be more important than bending of the ribs. Experience may indicate that for practical designs appreciable error is made by neglecting shear deflections, but the extension of the theory to take this into account has been left for possible future work. (See, however, ref. 5.)

The present method could be generalized to allow arbitrary chordwise bending. The generalization would involve replacing the double integral for the strain energy of the cover sheet by a double sum. The unknowns in the equations would be the values of w at many lattice points on the wing rather than the values of φ_k at a number of spanwise stations. The resulting theory would be similar to Williams' theory except that the ribs and spars would not be spread out to act with the sheet.

Some more precise treatment of chordwise bending would be necessary for a complete stress analysis of delta wings. The primary concern of this paper has been deflection analysis and, although the stresses can be estimated from the deflections, a double numerical differentiation is involved; consequently, the resulting stresses would be less accurate than the deflections. The parabolic or linear theory would give either constant or zero curvatures of the ribs; thus, the theory would seem to be of questionable value for determining stresses in these members. However, in reference 8 a comparison of experimental results with the predictions of the parabolic theory applied to a solid delta plate shows good agreement for the spanwise bending stresses.

MECHANICS OF APPLICATION

More theory and mathematical details have been given in the section entitled "Method of Analysis" than are necessary in actually setting up and solving a given problem. On the other hand, some troublesome points which may arise in the application of the method have been omitted so far. This section is intended to serve as a guide in applying the method and also to cover these troublesome details.

Steps in Setting up Matrices of Structural Properties

The following steps are used in setting up matrices of structural properties:

(1) Choose a coordinate system as in figure 5 with the origin on the center line.

(2) Choose a number of equally spaced stations along the half-span as in figure 6. There may be some "natural" number of stations dictated by the geometry of the structure but, in general, reasonable accuracy can be obtained with about eight stations. For convenience, the points of attachment of the wing to the fuselage should be made to fall on stations, if possible.

(3) Choose a neutral surface. The choice recommended in this paper is the locus of the principal centroidal axis of the chordwise sections. However, any discontinuities such as those due to cutouts and reinforcements should be smoothed out. Of course, in the process of finding the neutral surface the cross-section areas of spars and stringers, skin thicknesses, and so forth will be tabulated at the stations and half-stations. The effective cross-section areas of swept spars and stringers should be taken to be the actual cross-section areas (normal to their own axis) times a reduction factor of $\cos^3 \lambda_s$, where λ_s is the angle of sweep of the spar or stringer.

(4) Calculate tables of the coefficients $a_{k,n}$ and $a_{k,n+\frac{1}{2}}$ from the formulas

$$a_{k,n} = \int_{c_1}^{c_2} D_n x^k dx \qquad \begin{pmatrix} k = 0, 1, 2, 3, 4 \\ n = 0, 1, 2, \ldots N \end{pmatrix} \tag{84}$$

$$a_{k,n+\frac{1}{2}} = \int_{c_1}^{c_2} D_{n+\frac{1}{2}} x^k dx \qquad \begin{pmatrix} k = 0, 1, 2 \\ n = 0, 1, 2, \ldots N-1 \end{pmatrix} \tag{85}$$

where (see eqs. (5) and (7))

$$D = \frac{E}{1 - \mu^2}\left(t_u z_u^2 + t_l z_l^2\right) \tag{86}$$

These integrations can be done numerically if necessary. In case there are discontinuities in the cover stiffness D, some adjustment of the values of $a_{k,n}$ at the station nearest the discontinuity is necessary. Let the jump in a_k be Δ_k (positive if a_k increases across the discontinuity for increasing x) and let ϵd be the absolute value of the distance between the discontinuity and the nearest station m; then,

(a) If the discontinuity in a_k is a distance ϵd to the left of station m, replace $a_{k,m}$ by $a_{k,m} - \left(\frac{1}{2} - d\right)\Delta_k$.

(b) If the discontinuity in a_k is a distance ϵd to the right of station m, replace $a_{k,m}$ by $a_{k,m} + \left(\frac{1}{2} - d\right)\Delta_k$.

(c) If the discontinuity is at station m, use the mean value.

A similar rule holds for adjusting the values of the coefficients $a_{k,n+\frac{1}{2}}$. It is probably sufficiently accurate to obtain the values of the jumps Δ_k graphically.

(5) Calculate the coefficients $\beta_{s,n}$ for each spar and stringer at each station from the formula

$$\beta_{s,n} = \left(EI_s\right)_n \cos^3 \lambda_s \tag{87}$$

where I is computed for a cross section perpendicular to the spar about an axis lying in the neutral surface and λ_s is the angle of sweep of the sth spar (stringer); see figure 5.

(6) Tabulate the coefficients $\bar{\beta}_{s,n}$. The rules for finding the values of $\bar{\beta}$ in terms of β are

(a) If the left end of a spar (stringer) falls within $\frac{1}{2}\epsilon$ of station m, let ϵd_s be the signed distance from station m to the end of the spar; then,

$$\bar{\beta}_{s,m} = \left(\frac{1}{2} - d_s\right)\beta_{s,m} \tag{88}$$

(b) If the right end of a spar falls within $\frac{1}{2}\epsilon$ of station m', let $\epsilon d_s'$ be the signed distance from station m' to the end of the spar; then,

$$\bar{\beta}_{s,m'} = \left(\frac{1}{2} + d_s'\right)\beta_{s,m'} \tag{89}$$

(c) For $m < n < m'$,

$$\bar{\beta}_{s,n} = \beta_{s,n} \tag{90}$$

(d) If the spar crosses the center line,

$$\bar{\beta}_{s,0} = \frac{1}{2}\beta_{s,0} \qquad (91)$$

(e) Elsewhere,

$$\bar{\beta}_{s,n} = 0 \qquad (92)$$

Sometimes β_s will have to be extrapolated to obtain a value at a station point. For a light stringer, little error is committed by assuming the stringer to terminate at the nearest half-station on the outboard end; in this case, $\bar{\beta}$ is the same as β at any station (except at the center line) to which the stringer extends.

(7) Let $x_{s,n}$ be the distance from the y-axis to the sth spar (stringer) at station n. (See fig. 5.) Tabulate

(a) x_s for the unswept spars and stringers

(b) $x_{s,n}$ for the swept spars (stringers) including values for $x_{s,-1}$ if $\beta_{s,0} \neq 0$ and $x_{s,N+1}$ if $\beta_{s,N} \neq 0$. In any case, the values for x_s need only be tabulated for one station beyond the ends of the range for which $\beta_s \neq 0$. Values of x_s at stations off the span of the half-wing are to be obtained by extrapolation $\left(x_{s,-1} = 2x_{s,0} - x_{s,1}\right)$.

(8) Integrate the stiffnesses of the ribs and chordwise stiffeners across the chord to obtain a set of values of γ_r according to equation (13). For a rib on the center line, take γ_r to be one-half the value given by equation (13). For those ribs or stiffeners that fall between stations, the numbers d_r (where ϵd_r is the distance from the first station to the left of the rib) should be recorded.

All the necessary information has now been extracted from the design of the wing and arranged in tables of numbers.

(9) Set up the $\left[a_k\right]$, $\left[\bar{a}_k\right]$, and $\left[\hat{a}_0\right]$ matrices from the values of a_k at the stations and the $\left[a_k{}^*\right]$ matrices from the values of a_k at the half-stations according to equations (36) to (39) (symmetric) or (50) to (52) (antisymmetric).

(10) Set up the $\left[\bar{\beta}_s\right]$, $\left[X_s\right]$, and $\left[X_s^2\right]$ matrices for the swept spars and stringers. For spars or stringers not running the full length of the span, some of the columns in $\bar{\beta}_s$ will be zero, in which case the elements in the corresponding columns of $\left[X_s\right]$ and $\left[X_s^2\right]$ can be set equal to zero without affecting any subsequent result. For unswept spars and stringers, x_s = Constant; thus, $\left[X_s\right] = x_s\left[D_1\right]$ and $\left[X_s^2\right] = x_s^2\left[D_2\right]$. There is no need for setting up the $\left[X_s\right]$ and $\left[X_s^2\right]$ matrices in this case. A considerable saving in computational labor is afforded by including the effects of the unswept spars and stringers in with the skin (no approximations involved) in the following way: Set up the matrices $\left[\sum \bar{\beta}_s\right]$, $\left[\sum \bar{\beta}_s x_s\right]$, $\left[\sum \bar{\beta}_s x_s^2\right]$, $\left[\sum \bar{\beta}_s x_s^3\right]$, and $\left[\sum \bar{\beta}_s x_s^4\right]$ $\left(\text{which will be diagonal matrices of the same form as } \left[\bar{\beta}_s\right]\right)$. Add these matrices directly to $\left[a_0\right]$, $\left[a_1\right]$, $\left[a_2\right]$, $\left[a_3\right]$, and $\left[a_4\right]$, respectively $\left(\text{but not to } \left[\bar{a}_k\right], \left[a_k^*\right], \text{ or } \left[\hat{a}_0\right]\right)$.

(11) Set up the matrix $\left[\sum \Gamma_r\right]$ from the table of values of γ_r and d_r where each rib or stiffener contributes terms to the elements in the matrix $\left[\sum \Gamma_r\right]$ as follows:

If the rth rib (stiffener) is between stations m and m+1, a distance ϵd_r from station m, then the rth rib contributes the block of terms

$$\begin{matrix} \gamma_r\left(1 - d_r\right)^2 & \gamma_r d_r\left(1 - d_r\right) \\ \gamma_r d_r\left(1 - d_r\right) & \gamma_r d_r^2 \end{matrix}$$

where the upper left-hand element is in the (m+1)st row and column in the symmetric case and in the mth row and column in the antisymmetric case. For ribs at a station, $d_r = 0$. There is no contribution from a rib on the center line (station 0) in the antisymmetric case, and in this case those ribs between the center line and station 1 contribute only to the element in the first row and column of $\left[\sum \Gamma_r\right]$. In either the symmetric or antisymmetric case, $\left[\sum \Gamma_r\right]$ is a square matrix which is either diagonal or has numbers on the principal diagonal and the two diagonals next to it. In the symmetric case, $\left[\sum \Gamma_r\right]$ is an $(N+2)\times(N+2)$ matrix and in the antisymmetric case it is an $(N+1)\times(N+1)$ matrix. In either case, the last row and the last column are filled with zeros.

Calculation of $\left[A\right]^{-1}$ Matrix

There are two alternative procedures for obtaining the inverse of $\left[A\right]$. The first alternative involves setting up a matrix of the order of about $3N\times3N$ and inverting it. The second alternative involves setting up a matrix of the order of about $2N\times2N$ and inverting it and also setting up three auxiliary matrices and performing several matrix multiplications and additions subsequent to the inversion operation. The most economical choice between the two alternatives depends on the computing facilities available. In general, the first alternative involves the least nonautomatic computing and should be preferred when good high-speed computing equipment is available (provided N is not too large). The second alternative is also suitable for automatic computing, but involves the setting up of some additional matrices before the work is ready for the machine.

First alternative.- Using the results obtained thus far, form the matrices $\left[A_{ij}\right]$ according to equations (31). Note that the summations should be made only over swept spars and stringers if the effects of the unswept spars and stringers have been included in the $\left[a_k\right]$ matrices as stated in step (10) for setting up matrices of structural properties. Combine these matrices into the matrix $\left[A\right]$ according to equation (30). The next step is the inversion of the matrix $\left[A\right]$; however, in certain cases this matrix will be singular. The proper way in which to modify $\left[A\right]$ before inversion in certain of these cases is listed as follows:

(1) In case only two spars (which do not taper to zero) extend to a pointed tip, strike out the last row and column of $\left[A\right]$ $\left(\text{corresponds to setting } \varphi_{2,N+1} = 0\right)$.

(2) In case only one spar (which does not taper to zero) extends to a pointed tip, strike out the last row and column in $\left[A\right]$ and also the row and column in $\left[A\right]$ containing the last row and column of $\left[A_{11}\right]$ $\left(\text{corresponds to setting } \varphi_{1,N+1} = \varphi_{2,N+1} = 0\right)$.

(3) In case all spars taper to zero at a pointed tip, strike out the rows and columns in $\left[A\right]$ which contain the last rows and columns in $\left[A_{00}\right]$, $\left[A_{11}\right]$, and $\left[A_{22}\right]$ $\left(\text{corresponds to setting } \varphi_{0,N+1} = \varphi_{1,N+1} = \varphi_{2,N+1} = 0\right)$.

(4) If there is a carry-through bay with only two spars and no cover sheet or ribs, strike out the rows and columns in $\left[A\right]$ which contain the first few rows and columns in $\left[A_{22}\right]$. The number of rows and columns deleted is the number of stations within the carry-through bay (counting the station at the center line in the symmetric case but not counting it in the antisymmetric case). This corresponds to setting $\varphi_{2,0} = \varphi_{2,1} = \ldots = \varphi_{2,m} = 0$ where m is the last station within the carry-through bay. This case may occur in combination with the first three cases.

Second alternative.- Form the $\left[B_{ij}\right]$ matrices according to equations (60) and the $\left[C_1\right]$ and $\left[C_2\right]$ matrices according to equations (61). Combine these matrices to form $\left[B\right]$ given by equation (64) and $\left[C\right]$ given by equation (67). The next step is to obtain $\left[B\right]^{-1}$. Here again, however, the matrix may be singular. The appropriate modifications to $\left[B\right]$ if this is the case are as follows (where the cases are numbered as in the first alternative):

(1) Strike out the last row and column in $\left[B\right]$ and the last row of $\left[C\right]$.

(2) Strike out the last row and column in $\left[B\right]$ and the row and column in $\left[B\right]$ containing the last row and column of $\left[B_{11}\right]$. Also strike out the last row of $\left[C_1\right]$ and $\left[C_2\right]$.

(3) Strike out the last row and column of all matrices appearing in equations (60), (61), and (68) including $\left(\left[a_0\right] + \sum_s \left[\bar{\beta}_s\right]\right)$ and proceed as usual.

(4) Strike out the rows and columns in $\left[B\right]$ which contain the first few rows and columns in $\left[B_{22}\right]$. Strike out the first few rows in $\left[C_2\right]$. The number of rows or columns to delete is the same as in case (4) in the previous section.

After either the $\left[B\right]$ matrix or the modified $\left[B\right]$ matrix has been inverted, form the $\left[A\right]^{-1}$ matrix in equation (66) using the definition given by equation (65).

<u>The effect on the inverse of $\left[A\right]$ of a modification in the structure</u>.- If the $\left[A\right]^{-1}$ matrix has been computed and some change in the structure is made that affects only a few of the elements in $\left[A\right]$, then the work of obtaining the new inverse matrix can be greatly reduced by making use of a method given in reference 9. The new inverse is obtained exactly whether or not the changes in the elements of $\left[A\right]$ are small. This method would be particularly useful for finding the effect of modifying one or two ribs, or a spar over a small portion of the span.

Calculation of the $\left[g\right]$ Matrix

According to equation (66), which is

$$\left[\varphi\right] = \epsilon^3 \left[A\right]^{-1} \left[p\right] \tag{93}$$

there are rows and columns in $\left[A\right]^{-1}$ corresponding to $\varphi_{k,N+1}$ and $p_{k,N+1}$. If these rows and columns are not already missing in $\left[A\right]^{-1}$

by virtue of their deletion under case (1), (2), (3), or (4) of the last section, they must now be eliminated. One way to locate the rows in question is to write down the column matrix $[\varphi]$. There are as many rows in $[\varphi]$ as there are in $[A]^{-1}$. Thus, if no rows and columns in $[A]$ have been deleted, then the rows to be deleted in $[A]^{-1}$ are in the position marked with an asterisk. The corresponding columns in $[A]^{-1}$ are also deleted and the result is a symmetric matrix.

$$
\begin{array}{cc}
\text{Symmetrical case} & \text{Antisymmetrical case} \\
\begin{array}{c}
\begin{bmatrix}
\varphi_{0,1} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{0,N} \\ \varphi_{0,N+1} \\ \varphi_{1,1} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{1,N} \\ \varphi_{1,N+1} \\ \varphi_{2,0} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{2,N} \\ \varphi_{2,N+1}
\end{bmatrix}
\begin{matrix}
\\ \\ \\ \\ \\ * \\ \\ \\ \\ \\ \\ * \\ \\ \\ \\ \\ \\ *
\end{matrix}
\end{array}
&
\begin{array}{c}
\begin{bmatrix}
\varphi_{0,2} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{0,N} \\ \varphi_{0,N+1} \\ \varphi_{1,1} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{1,N} \\ \varphi_{1,N+1} \\ \varphi_{2,1} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{2,N} \\ \varphi_{2,N+1}
\end{bmatrix}
\begin{matrix}
\\ \\ \\ \\ \\ * \\ \\ \\ \\ \\ \\ * \\ \\ \\ \\ \\ \\ *
\end{matrix}
\end{array}
\end{array}
$$

In addition to the foregoing modifications, two rows and columns of zeros should be inserted in the $\left[A\right]^{-1}$ matrix in the positions marked with an asterisk in the symmetric case. In the antisymmetric case, $\left[A\right]^{-1}$ is to be bordered on the top and left by a row and column of zeros as indicated.

$$\begin{array}{cc}
\text{Symmetrical case} & \text{Antisymmetrical case} \\
\begin{array}{c}\left[\begin{array}{c} 0 \\ \varphi_{0,1} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{0,N} \\ 0 \\ \varphi_{1,1} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{1,N} \\ \varphi_{2,0} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{2,N} \end{array}\right]\end{array}\begin{array}{c} * \\ \\ \\ \\ \\ \\ * \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}
&
\left[\begin{array}{c} 0 \\ \varphi_{0,2} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{0,N} \\ \varphi_{1,1} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{1,N} \\ \varphi_{2,1} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{2,N} \end{array}\right]\begin{array}{c} * \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}
\end{array}$$

Finally the modified $\left[A\right]^{-1}$ matrix is multiplied by ϵ^3 and the result called $\left[g\right]$ as in equation (69).

Influence Coefficient Matrix

Choose a set of reference points on the plan form of the wing for which influence coefficients are desired (see fig. 7). Include among them the points of attachment of the wing to the fuselage. These reference points must be at the spanwise stations, but there may be any number of them along a chord at a given station. Construct the matrix $\left[H\right]$ (given by eq. (75) in the symmetric case or by eq. (76) in the antisymmetric case) from the values of $\xi_{n,m}$ which are the distances from the y-axis to the mth reference point on the chord at station n. Calculate the influence-coefficient matrix $\left[G\right]$ from equation (79). The influence-coefficient matrix for the wing supported at an arbitrary number of flexible supports may be found by applying the method given in appendix A.

CONCLUDING REMARKS

A method for obtaining influence coefficients for thin low-aspect-ratio wings has been presented. The application of the method has been organized as much as possible into a routine procedure.

The technique for arriving at difference equations in matrix form directly (rather than via differential equations) has afforded considerable economy of thought particularly in the treatment of boundary conditions at the wing tip and in the handling of discontinuities.

The development (after eq. (69)) has been limited to finding influence coefficients; however, all problems are not necessarily most conveniently handled by means of influence coefficients. For instance, in the problem of finding natural modes and frequencies, it is convenient to introduce generalized inertia loads on the right-hand side of equation (69) in place of the generalized loads matrix $\left[p\right]$. The proper matrix form for the inertia terms can be deduced from the expression for the kinetic energy in discrete form. The frequencies and (generalized) modes can be found directly from the matrix equations without any need for finding influence coefficients.

Theoretical influence coefficients obtained by the present method have not yet been checked experimentally. Perhaps, for practical purposes, a close check on influence coefficients is too severe a criterion for the usefulness of the theory. That is, there may be greater discrepancies between predicted and experimental deflections due to a concentrated load than there would be if the load were more uniformly distributed. In case of serious discrepancies between predicted and experimental results, neglect of the effect of transverse shear deflection in the theory should probably be the first assumption to question.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., January 31, 1956.

APPENDIX A

INFLUENCE COEFFICIENTS FOR THE WING ATTACHED TO THE FUSELAGE

This appendix will be concerned with the problem of determining a set of influence coefficients for the wing which takes into account the fact that the wing is attached to a flexible fuselage. These influence coefficients will be determined by appropriately modifying the set of influence coefficients $\left[G\right]$ appearing in equation (79). For present purposes, the set of influence coefficients $\left[G\right]$ must include those coefficients which refer to the points of attachment with the fuselage. The only information about the fuselage necessary for the solution of the problem is a set of influence coefficients for those points of the fuselage where the wing is attached. The coefficients for the fuselage need not necessarily be obtained from as refined an analysis of the fuselage as has been carried out for the wing. For example, they might be obtained by approximating the fuselage by a simple beam in the symmetric case or by a torsion box in the antisymmetric case, or they may even be taken to be zero corresponding to the crude assumption of a rigid fuselage. The influence coefficients for the fuselage are supposed to be obtained by assuming the fuselage to be supported in such a way as to prevent rigid-body motions. However, the reactions at these supports must vanish when the fuselage is subjected to any self-equilibrated set of loads.

Mathematical Derivation in the Symmetric Case

Let the influence coefficients for the fuselage which refer only to the points of attachment of the wing be arranged in the $m \times m$ matrix, $\left[J\right]_S$ (S for symmetric deformations), where m is the number of attachment points. If necessary, rearrange the matrix $\left[G\right]_S$ so that the influence coefficients which refer only to the points of attachment appear in the upper left-hand corner in the same order as the corresponding coefficients in $\left[J\right]_S$.

Let the reactions at the m supports be written as the column matrix $\left[R\right]$ and let $\left[R\right]_B$ be $\left[R\right]$ augmented by zeros to make up a column of M elements where M is the total number of reference points. Let the external loads be written as the column matrix $\left[P\right]$ with M rows. Let $\left[\overline{W}\right]$ be a column matrix of m rows whose elements are the deflections at the supports and let W be the column matrix of deflections at all M reference points on the wing where a given reference point is at the point (ξ,η). Furthermore, define the following matrices:

$\left[\bar{G}\right]_S$ matrix of the first m rows of $\left[G\right]_S$

$\left[\bar{1}\right]$ column of m ones

$\left[1\right]$ column of M ones

$\left[\bar{\xi}\right]$ column of ξ's at the first m reference points (supports)

$\left[\xi\right]$ column of ξ's at all M reference points

For symmetric loading the deflection of the wing is given by

$$\left[W\right] = \left[G\right]_S\left(\left[P\right] + \left[R\right]_B\right) + \zeta\left[1\right] + \theta\left[\xi\right] \tag{A1}$$

The last two terms on the right are the symmetric rigid-body displacements, namely, vertical translation and pitch. The deflection at the supports only is

$$\left[\bar{W}\right] = \left[\bar{G}\right]_S\left[P\right] + \left[\hat{G}\right]_S\left[R\right] + \zeta\left[\bar{1}\right] + \theta\left[\bar{\xi}\right] = -\left[J\right]_S\left[R\right] \tag{A2}$$

where $\left[\hat{G}\right]_S$ is the m×m matrix in the upper left-hand corner of $\left[G\right]_S$. The last term on the right is the deflection of the supports calculated by applying the (equal and opposite) reaction forces on the fuselage. From equation (A2),

$$-\left(\left[J\right]_S + \left[\hat{G}\right]_S\right)\left[R\right] = -\left[T\right]\left[R\right] = \left[\bar{G}\right]_S\left[P\right] + \zeta\left[\bar{1}\right] + \theta\left[\bar{\xi}\right] \tag{A3}$$

where now $\left[T\right] = \left[J\right]_S + \left[\hat{G}\right]_S$ is a symmetric m×m matrix. The inverse of $\left[T\right]$ must be computed, but it is ordinarily a low-order matrix. From equation (A3), the following equation is obtained:

$$-[R] = [T]^{-1}[\bar{G}]_S[P] + \zeta[T]^{-1}[\bar{1}] + \theta[T]^{-1}[\bar{\xi}] \tag{A4}$$

Inasmuch as the wing is in equilibrium under the loads $[P]$ and $[R]$, the resultant vertical force and pitching moment due to these loads must vanish; this condition yields the following two equations:

$$[\bar{1}]'[R] + [1]'[P] = 0 \tag{A5}$$

$$[\bar{\xi}]'[R] + [\xi]'[P] = 0 \tag{A6}$$

where primes denote the transpose of a matrix. Combining equations (A4) to (A6) gives the following two equations for the determination of ζ and θ:

$$[1]'[P] = [\bar{1}]'[T]^{-1}[\bar{G}]_S[P] + \zeta[\bar{1}]'[T]^{-1}[\bar{1}] + \theta[\bar{1}]'[T]^{-1}[\bar{\xi}] \tag{A7}$$

$$[\xi]'[P] = [\bar{\xi}]'[T]^{-1}[\bar{G}]_S[P] + \zeta[\bar{\xi}]'[T]^{-1}[\bar{1}] + \theta[\bar{\xi}]'[T]^{-1}[\bar{\xi}] \tag{A8}$$

or

$$[\bar{1}]'[T]^{-1}[\bar{1}]\zeta + [\bar{1}]'[T]^{-1}[\bar{\xi}]\theta = \left([1]' - [\bar{1}]'[T]^{-1}[\bar{G}]_S\right)[P] \tag{A9}$$

$$[\bar{\xi}]'[T]^{-1}[\bar{1}]\zeta + [\bar{\xi}]'[T]^{-1}[\bar{\xi}]\theta = \left([\xi]' - [\bar{\xi}]'[T]^{-1}[\bar{G}]_S\right)[P] \tag{A10}$$

These equations must now be solved for ζ and θ. The following (scalar) numbers have to be computed:

$$\left.\begin{aligned}
\Delta &= \left([\bar{1}]'[T]^{-1}[\bar{1}]\right)\left([\bar{\xi}]'[T]^{-1}[\bar{\xi}]\right) - \left([\bar{\xi}]'[T]^{-1}[\bar{1}]\right)^2 \\
\alpha &= \left([\bar{\xi}]'[T]^{-1}[\bar{\xi}]\right)\Big/\Delta \\
\beta &= \left([\bar{1}]'[T]^{-1}[\bar{\xi}]\right)\Big/\Delta \\
\gamma &= \left([\bar{1}]'[T]^{-1}[\bar{1}]\right)\Big/\Delta
\end{aligned}\right\} \tag{A11}$$

Solving for ζ and θ gives

$$\zeta = \alpha\left([1]' - [\bar{1}]'[T]^{-1}[\bar{G}]_S\right)[P] - \beta\left([\xi]' - [\bar{\xi}]'[T]^{-1}[\bar{G}]_S\right)[P] \tag{A12}$$

$$\theta = \gamma\left([\xi]' - [\bar{\xi}]'[T]^{-1}[\bar{G}]_S\right)[P] - \beta\left([1]' - [\bar{1}]'[T]^{-1}[\bar{G}]_S\right)[P] \tag{A13}$$

With the help of the matrix $[T]_B^{-1}$ (which is $[T]^{-1}$ bordered by zeros to make up an MXM matrix), these equations may be written:

$$\zeta = \left(\alpha[1]' - \beta[\xi]'\right)\left([I] - [T]_B^{-1}[G]_S\right)[P] \tag{A14}$$

$$\theta = \left(\gamma[\xi]' - \beta[1]'\right)\left([I] - [T]_B^{-1}[G]_S\right)[P] \tag{A15}$$

where $[I]$ is the unit matrix. Now that ζ and θ are known, $[R]_B$ may be found from equation (A4) and is as follows:

$$\begin{aligned}
[R]_B = -[T]_B^{-1}\Big\{[G]_S &+ \left(\alpha[1][1]' - \beta[1][\xi]'\right)\left([I] - [T]_B^{-1}[G]_S\right) + \\
&\left(\gamma[\xi][\xi]' - \beta[\xi][1]'\right)\left([I] - [T]_B^{-1}[G]_S\right)\Big\}[P]
\end{aligned} \tag{A16}$$

Introducing these results into equation (A1) gives

$$\left[W\right] = \left(\left[I\right] - \left[G\right]_S\left[T\right]_B^{-1}\right)\left\{\left[G\right]_S + \left(\alpha\left[1\right]\left[1\right]' - \beta\left[1\right]\left[\xi\right]' - \beta\left[\xi\right]\left[1\right]' + \gamma\left[\xi\right]\left[\xi\right]'\right)\left(\left[I\right] - \left[T\right]_B^{-1}\left[G\right]_S\right)\right\}\left[P\right] \tag{A17}$$

or

$$\left[W\right] = \left[K\right]_S\left[P\right] \tag{A18}$$

The square symmetric matrix $\left[K\right]_S$ whose definition is obvious from equations (A17) and (A18) is the required modified set of influence coefficients for the wing.

Results in the Antisymmetric Case

The modified matrix of influence coefficients in the antisymmetric case is simpler because only one rigid-body motion is involved, namely a roll. In this case,

$$\left[K\right]_A = \left(\left[I\right] - \left[G\right]_A\left[T\right]_B^{-1}\right)\left\{\left[G\right]_A + \alpha\left[\overline{\eta}\right]\left[\eta\right]'\left(\left[I\right] - \left[T\right]_B^{-1}\left[G\right]_A\right)\right\} \tag{A19}$$

where now $\left[\overline{\eta}\right]$ is the column of η's at the first m reference points and $\left[\eta\right]$ is the column of η's at all M reference points. Also

$$\left[T\right] = \left[J\right]_A + \left[\hat{G}\right]_A \tag{A20}$$

and

$$\frac{1}{\alpha} = \left[\overline{\eta}\right]'\left[T\right]^{-1}\left[\overline{\eta}\right] \tag{A21}$$

Exceptional Support Conditions

In general the $[T]$ matrix in equation (A3) will be nonsingular, but in certain special cases it will be singular and some modification of the method will be necessary. These cases are listed as follows:

(1) The $[\hat{G}]$ matrix will be singular if there are more than three supports along any one chord in the parabolic theory (the number is two in the linear theory) because the influence coefficients for four or more points along a chord which deforms into a parabola are linearly dependent. If the supports are fixed $\left(\text{so } [J] = 0\right)$, then $[T]$ will be singular. There is no remedy for this situation other than to reduce the number of supports - and this reduction does not result in any loss of generality within the framework of parabolic theory.

(2) In the symmetric case, if the origin is to be a fixed support, then exclude the origin from the list of reference points. Since the origin is fixed, ζ will be zero and one equation must be deleted from the set of equations used to determine all the unknowns; this will be equation (A5). The effect on the result, namely $[K]_S$ in equation (A18), is to make

$$\alpha = \beta = 0 \qquad \frac{1}{\gamma} = [\bar{\xi}]'[T]^{-1}[\bar{\xi}]$$

(3) In the symmetric case of the linear theory, if any point along the center line is to be fixed, then exclude this point from the list of reference points. For purposes of forming the $[\xi]$ and $[\bar{\xi}]$ matrices, choose a new y-axis passing through this point and set $\zeta = 0$. Again the effect on the result is to make

$$\alpha = \beta = 0 \qquad \frac{1}{\gamma} = [\bar{\xi}]'[T]^{-1}[\bar{\xi}]$$

(4) In the antisymmetric case, if the points $(x=0, y=\pm\epsilon)$ are to be fixed, then exclude these points from the list of reference points and set $\alpha = 0$ in equation (A19).

Miscellaneous Remarks

In cases where some of the points of attachment cannot conveniently be made to fall on a station, influence coefficients for these points can be obtained by interpolation from surrounding reference points.

The effect of a couple applied at a point of attachment can be roughly approximated by replacing the couple by equal and opposite forces at nearby reference points.

Clearly, a set of influence coefficients for the whole fuselage could be modified in exactly the same way as in the previous section in order to take account of the presence of the wing. From there it would be only a step to obtain an overall set of influence coefficients for the wing-fuselage combination, the connecting link being furnished by the expressions for the reactions at the points of attachment. Since the required matrix manipulations are quite elementary, and the present paper is concerned primarily with the wing, the details will not be given herein.

A set of influence coefficients obtained experimentally could be modified by the method given in this appendix to obtain a modified set appropriate to some other system of supports than that used during the test. The only provision is that the support system used in the test be such that the reactions at the supports would vanish whenever the structure is subjected to a self-equilibrated set of loads. Fixing the structure at three points is one possibility.

APPENDIX B

EXAMPLE DERIVATION OF A MATRIX EQUATION

In this appendix the matrix form of the quantity $\frac{\partial \Pi_s}{\partial \varphi_{k,n}}$ is derived. This quantity is the term contributed by a spar to equations (28). The derivation will be made for the symmetric case.

From equation (22)

$$\Pi_s = \frac{1}{2\epsilon^3}\left(\bar{\beta}_{s,0}\psi_{s,0}^{\ 2} + \bar{\beta}_{s,1}\psi_{s,1}^{\ 2} + \ldots + \bar{\beta}_{s,N}\psi_{s,N}^{\ 2}\right) \tag{B1}$$

where

$$\psi_{s,n} = \chi_{s,n-1} - 2\chi_{s,n} + \chi_{s,n+1} \tag{B2}$$

$$\chi_{s,n} = \varphi_{0,n} + x_{s,n}\varphi_{1,n} + x_{s,n}^{\ 2}\varphi_{2,n} \tag{B3}$$

The boundary conditions are (from eq. (25))

$$\varphi_{0,0} = \varphi_{1,0} = 0 \qquad \varphi_{0,-1} = \varphi_{0,1} \qquad \varphi_{1,-1} = \varphi_{1,1} \qquad \varphi_{2,-1} = \varphi_{2,1} \tag{B4}$$

The first few of equations (B3) read

$$\left.\begin{aligned}
\chi_{s,-1} &= \varphi_{0,1} + x_{s,-1}\varphi_{1,1} + x_{s,-1}^{\ 2}\varphi_{2,1} \\
\chi_{s,0} &= x_{s,0}^{\ 2}\varphi_{2,0} \\
\chi_{s,1} &= \varphi_{0,1} + x_{s,1}\varphi_{1,1} + x_{s,1}^{\ 2}\varphi_{2,1} \\
\chi_{s,2} &= \varphi_{0,2} + x_{s,2}\varphi_{1,2} + x_{s,2}^{\ 2}\varphi_{2,2}
\end{aligned}\right\} \tag{B5}$$

The first few of equations (B2) read

$$\left.\begin{aligned}
\psi_{s,0} &= \chi_{s,-1} - 2\chi_{s,0} + \chi_{s,1} \\
\psi_{s,1} &= \chi_{s,0} - 2\chi_{s,1} + \chi_{s,2} \\
\psi_{s,2} &= \chi_{s,1} - 2\chi_{s,2} + \chi_{s,3}
\end{aligned}\right\} \tag{B6}$$

The first two equations of $\frac{\partial \Pi_s}{\partial \varphi_{0,n}}$ read (with the use of eqs. (B5) and (B6))

$$\left.\begin{aligned}
\frac{\partial \Pi_s}{\partial \varphi_{0,1}} &= \frac{1}{\epsilon^3}\left(2\bar{\beta}_{s,0}\psi_{s,0} - 2\bar{\beta}_{s,1}\psi_{s,1} + \bar{\beta}_{s,2}\psi_{s,2}\right) \\
\frac{\partial \Pi_s}{\partial \varphi_{0,2}} &= \frac{1}{\epsilon^3}\left(\bar{\beta}_{s,1}\psi_{s,1} - 2\bar{\beta}_{s,2}\psi_{s,2} + \bar{\beta}_{s,3}\psi_{s,3}\right)
\end{aligned}\right\} \tag{B7}$$

The first two equations of $\frac{\partial \Pi_s}{\partial \varphi_{1,n}}$ read

$$\left.\begin{aligned}
\frac{\partial \Pi_s}{\partial \varphi_{1,1}} &= \frac{1}{\epsilon^3}\left[\left(x_{s,-1} + x_{s,1}\right)\bar{\beta}_{s,0}\psi_{s,0} - 2x_{s,1}\bar{\beta}_{s,1}\psi_{s,1} + x_{s,1}\bar{\beta}_{s,2}\psi_{s,2}\right] \\
\frac{\partial \Pi_s}{\partial \varphi_{1,2}} &= \frac{1}{\epsilon^3}\left(x_{s,2}\bar{\beta}_{s,1}\psi_{s,1} - 2x_{s,2}\bar{\beta}_{s,2}\psi_{s,2} + x_{s,2}\bar{\beta}_{s,3}\psi_{s,3}\right)
\end{aligned}\right\} \tag{B8}$$

The first two equations of $\frac{\partial \Pi_s}{\partial \varphi_{2,n}}$ read

$$\left.\begin{aligned}
\frac{\partial \Pi_s}{\partial \varphi_{2,0}} &= \frac{1}{\epsilon^3}\left(-2x_{s,0}{}^2\bar{\beta}_{s,0}\psi_{s,0} + x_{s,0}{}^2\bar{\beta}_{s,1}\psi_{s,1}\right) \\
\frac{\partial \Pi_s}{\partial \varphi_{2,1}} &= \frac{1}{\epsilon^3}\left[\left(x_{s,-1}{}^2 + x_{s,1}{}^2\right)\bar{\beta}_{s,0}\psi_{s,0} - 2x_{s,1}{}^2\bar{\beta}_{s,1}\psi_{s,1} + x_{s,1}{}^2\bar{\beta}_{s,2}\psi_{s,2}\right]
\end{aligned}\right\} \tag{B9}$$

From equations (B7), the first two rows of the matrix $\left[\frac{\partial \Pi_s}{\partial \varphi_{0,n}}\right]$ may be written as follows:

$$\begin{bmatrix} \frac{\partial \Pi_s}{\partial \varphi_{0,1}} \\ \frac{\partial \Pi_s}{\partial \varphi_{0,2}} \\ \vdots \end{bmatrix} = \frac{1}{\epsilon^3} \begin{bmatrix} 2 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & & & \cdots \end{bmatrix} \begin{bmatrix} \bar{\beta}_{s,0} & & \\ & \bar{\beta}_{s,1} & \\ & & \cdots \end{bmatrix} \begin{bmatrix} \psi_{s,0} \\ \psi_{s,1} \\ \vdots \end{bmatrix} \tag{B10}$$

Similarly, from equations (B8),

$$\begin{bmatrix} \frac{\partial \Pi_s}{\partial \varphi_{1,1}} \\ \frac{\partial \Pi_s}{\partial \varphi_{1,2}} \\ \vdots \end{bmatrix} = \frac{1}{\epsilon^3} \begin{bmatrix} x_{s,-1} + x_{s,1} & -2x_{s,1} & x_{s,1} & & \\ & x_{s,2} & -2x_{s,2} & x_{s,2} & \\ & & & & \cdots \end{bmatrix} \begin{bmatrix} \bar{\beta}_{s,0} & & \\ & \bar{\beta}_{s,1} & \\ & & \cdots \end{bmatrix} \begin{bmatrix} \psi_{s,0} \\ \psi_{s,1} \\ \vdots \end{bmatrix} \tag{B11}$$

and, from equations (B9),

$$\begin{bmatrix} \frac{\partial \Pi_s}{\partial \varphi_{2,0}} \\ \frac{\partial \Pi_s}{\partial \varphi_{2,1}} \\ \vdots \end{bmatrix} = \frac{1}{\epsilon^3} \begin{bmatrix} -2x_{s,0}^2 & x_{s,0}^2 & & & \\ x_{s,-1}^2 + x_{s,1}^2 & -2x_{s,1}^2 & x_{s,1}^2 & & \\ & x_{s,2}^2 & -2x_{s,2}^2 & x_{s,2}^2 & \\ & & & & \cdots \end{bmatrix} \begin{bmatrix} \bar{\beta}_{s,0} & & \\ & \bar{\beta}_{s,1} & \\ & & \cdots \end{bmatrix} \begin{bmatrix} \psi_{s,0} \\ \psi_{s,1} \\ \vdots \end{bmatrix} \tag{B12}$$

Obviously, if more rows were written the 1 -2 1 pattern would be followed until the last few rows, where the vanishing of $\bar{\beta}_{s,n}$ for $n > N$ changes the pattern. The first few rows of $\left[\psi_{s,n}\right]$ may be written

$$\begin{bmatrix} \psi_{s,0} \\ \psi_{s,1} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} 2 & & & \\ -2 & 1 & & \\ 1 & -2 & 1 & \\ & & & \cdots \end{bmatrix} \begin{bmatrix} \varphi_{0,1} \\ \varphi_{0,2} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} + \begin{bmatrix} x_{s,-1} + x_{s,1} & & & \\ -2x_{s,1} & x_{s,2} & & \\ x_{s,1} & -2x_{s,2} & x_{s,3} & \\ & & & \cdots \end{bmatrix} \begin{bmatrix} \varphi_{1,1} \\ \varphi_{1,2} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} +$$

$$\begin{bmatrix} -2x_{s,0}^2 & x_{s,-1}^2 + x_{s,1}^2 & & & \\ x_{s,0}^2 & -2x_{s,1}^2 & x_{s,2}^2 & & \\ & x_{s,1}^2 & -2x_{s,2}^2 & x_{s,3}^2 & \\ & & & & \cdots \end{bmatrix} \begin{bmatrix} \varphi_{2,0} \\ \varphi_{2,1} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \tag{B13}$$

The three rectangular matrices appearing in equation (B13) are the transposes of the three rectangular matrices appearing in equations (B10), (B11), and (B12). The form of these matrices in their lower right-hand corners can be obtained in a way similar to that which was just used to obtain the form in the upper left-hand corner. Equations (B10), (B11), and (B12) can be written compactly as follows:

$$\left[\frac{\partial \Pi_s}{\partial \varphi_0}\right] = \frac{1}{\epsilon^3}\left[D_0\right]\left[\bar{\beta}_s\right]\left(\left[D_0\right]'\left[\varphi_0\right] + \left[X_s\right]'\left[\varphi_1\right] + \left[X_s^2\right]'\left[\varphi_2\right]\right) \tag{B14}$$

$$\left[\frac{\partial \Pi_s}{\partial \varphi_1}\right] = \frac{1}{\epsilon^3}\left[X_s\right]\left[\bar{\beta}_s\right]\left(\left[D_0\right]'\left[\varphi_0\right] + \left[X_s\right]'\left[\varphi_1\right] + \left[X_s^2\right]'\left[\varphi_2\right]\right) \tag{B15}$$

$$\left[\frac{\partial \Pi_s}{\partial \varphi_2}\right] = \frac{1}{\epsilon^3}\left[X_s^2\right]\left[\bar{\beta}_s\right]\left(\left[D_0\right]'\left[\varphi_0\right] + \left[X_s\right]'\left[\varphi_1\right] + \left[X_s^2\right]'\left[\varphi_2\right]\right) \tag{B16}$$

These equations can be further combined to give

$$\begin{bmatrix} \dfrac{\partial \Pi_s}{\partial \varphi_0} \\ \dfrac{\partial \Pi_s}{\partial \varphi_1} \\ \dfrac{\partial \Pi_s}{\partial \varphi_2} \end{bmatrix} = \frac{1}{\epsilon^3} \begin{bmatrix} [D_0][\bar{\beta}_s][D_0]' & [D_0][\bar{\beta}_s][X_s]' & [D_0][\bar{\beta}_s][X_s^2]' \\ [X_s][\bar{\beta}_s][D_0]' & [X_s][\bar{\beta}_s][X_s]' & [X_s][\bar{\beta}_s][X_s^2]' \\ [X_s^2][\bar{\beta}_s][D_0]' & [X_s^2][\bar{\beta}_s][X_s]' & [X_s^2][\bar{\beta}_s][X_s^2]' \end{bmatrix} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{bmatrix} \tag{B17}$$

A comparison of equation (B17) with equation (30) will show how the sth spar contributes to the $\left[A_{ij}\right]$ matrices defined in equations (31). Similar derivations were made for the covers and ribs but because the algebra is long and tedious only the results are given in the text.

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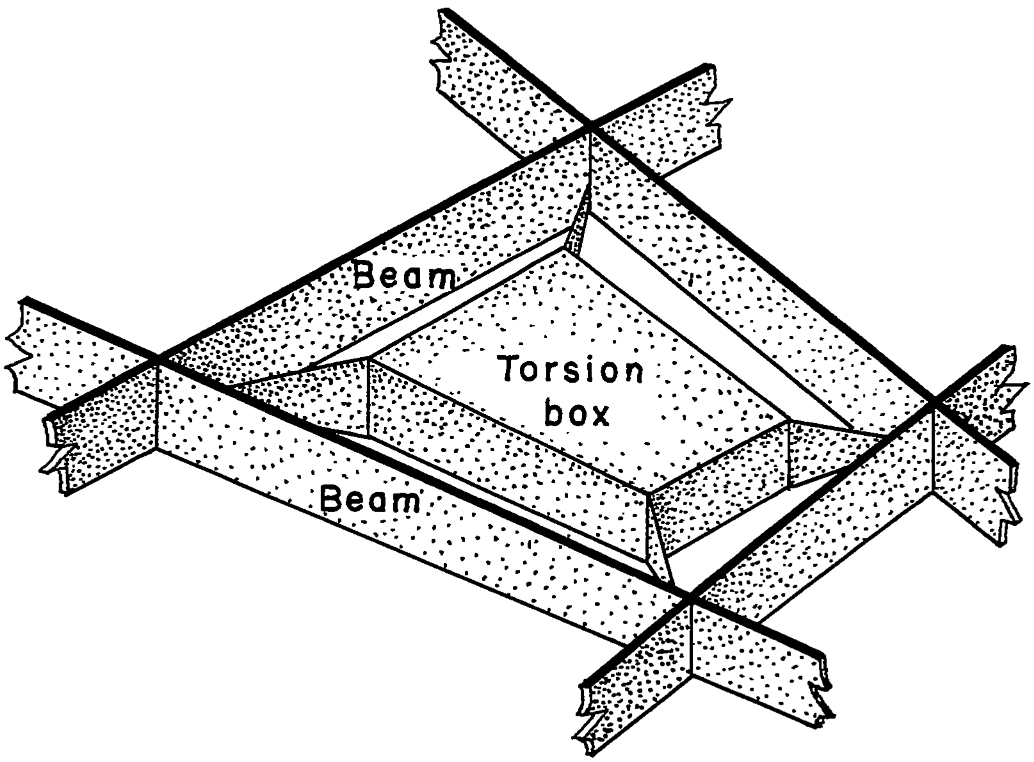

Figure 1.- Levy's idealization of a portion of a wing between spars and ribs.

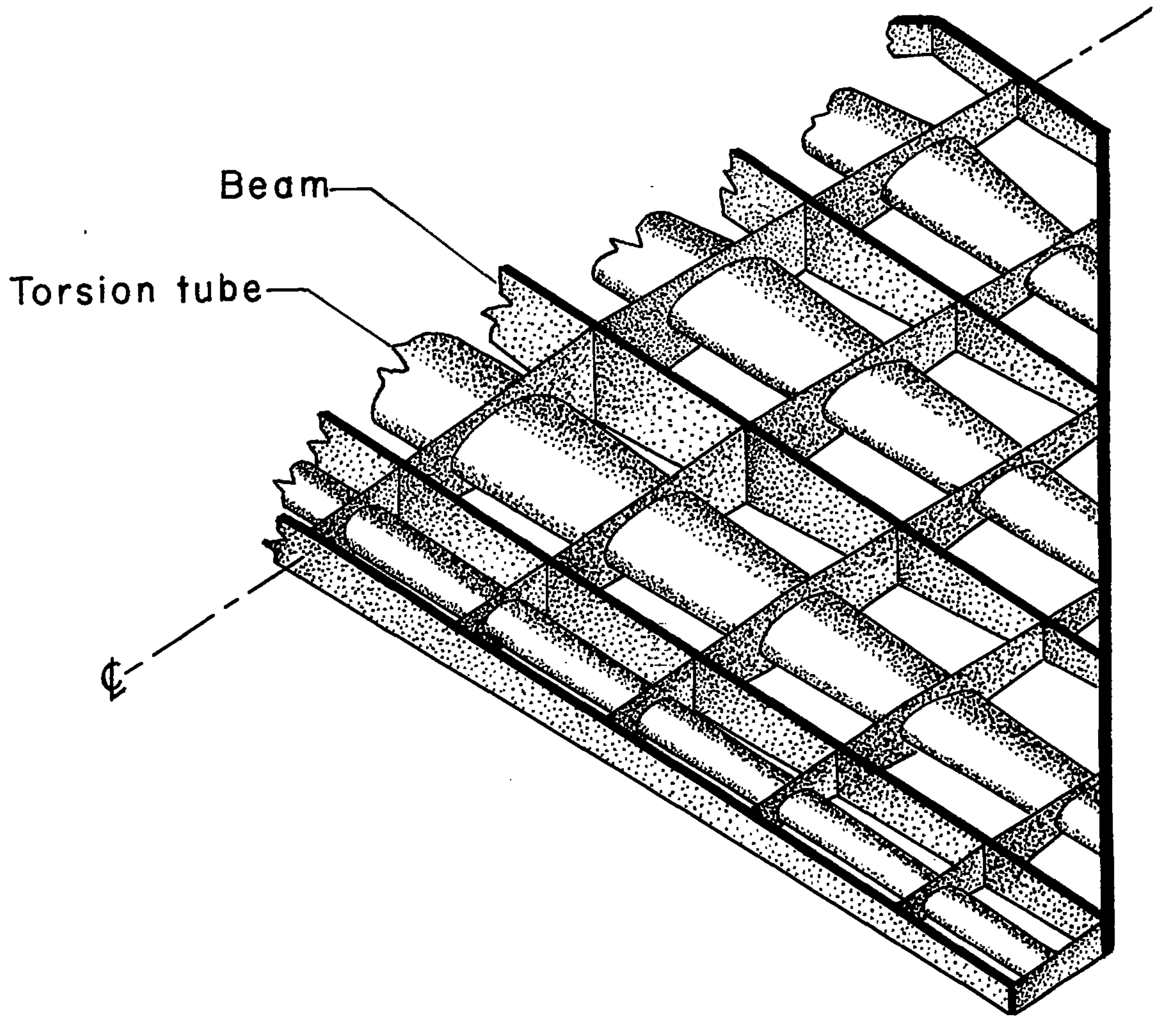

Figure 2.- Schuerch's idealization of a wing as a bundle of torsion tubes and beams tied together by rigid ribs.

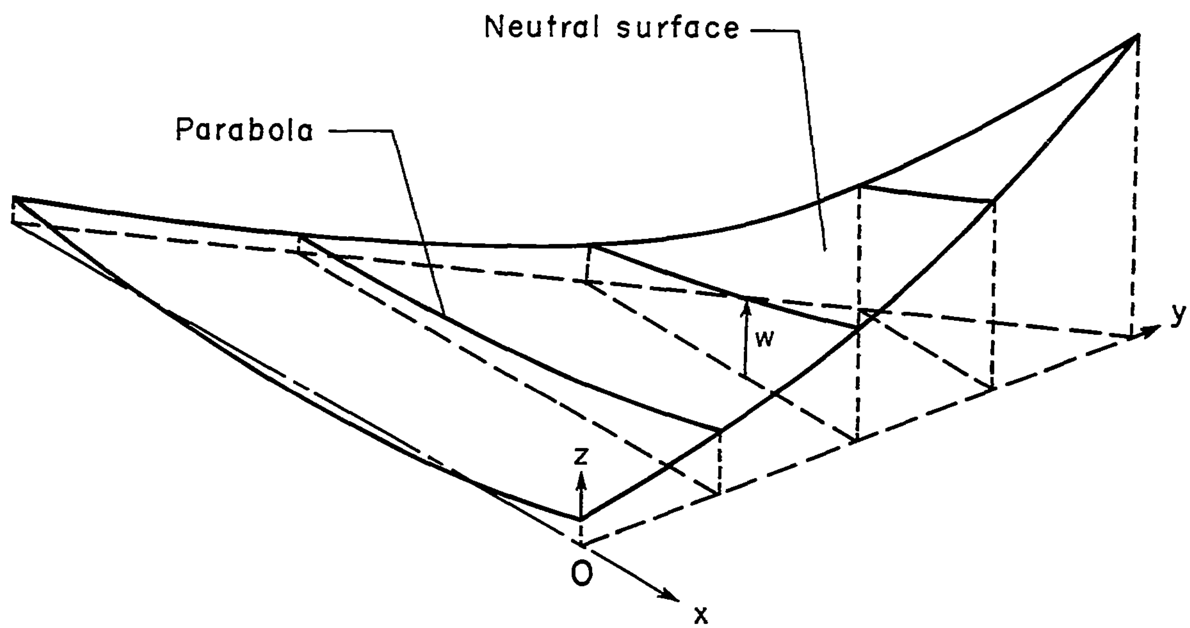

Figure 3.- Diagram of neutral surface showing assumed parabolic chordwise deflection shape.

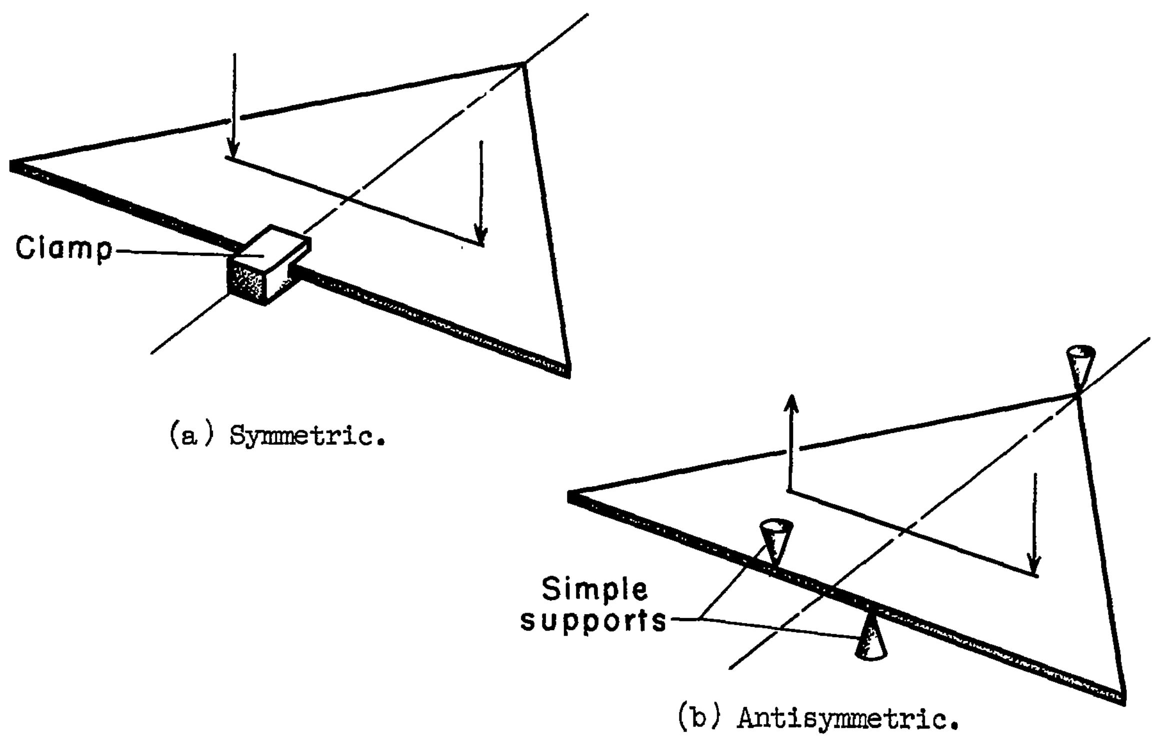

Figure 4.- Symmetric and antisymmetric support conditions considered.

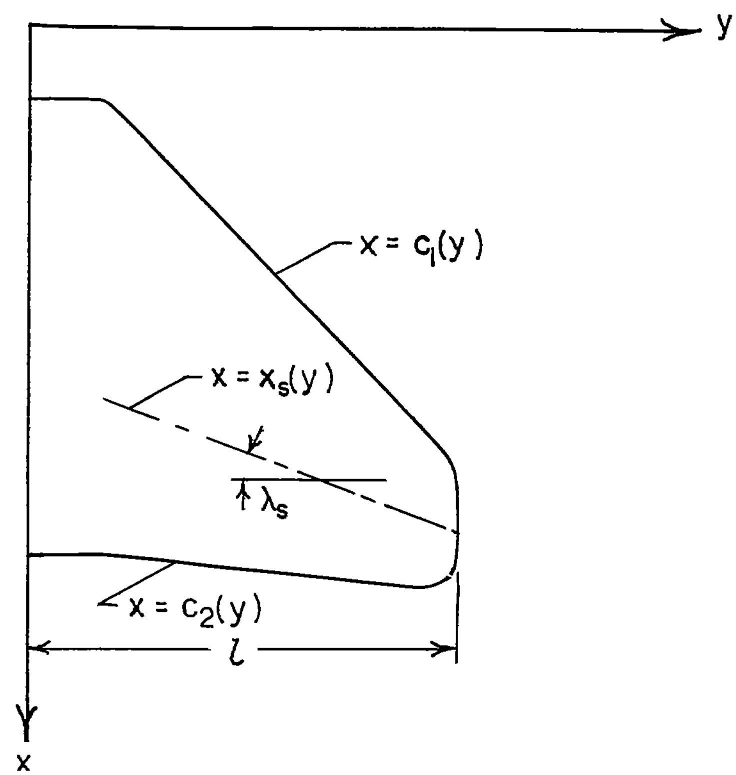

Figure 5.- Coordinate system and location of typical (sth) spar or stringer on plan form of half-wing.

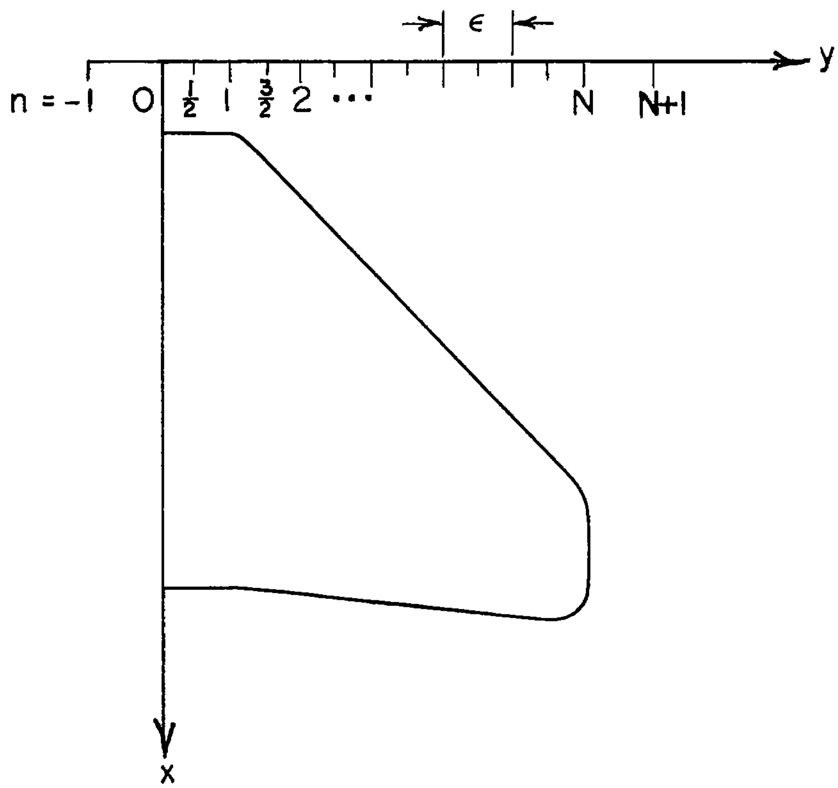

Figure 6.- Location of stations on plan form of half-wing.

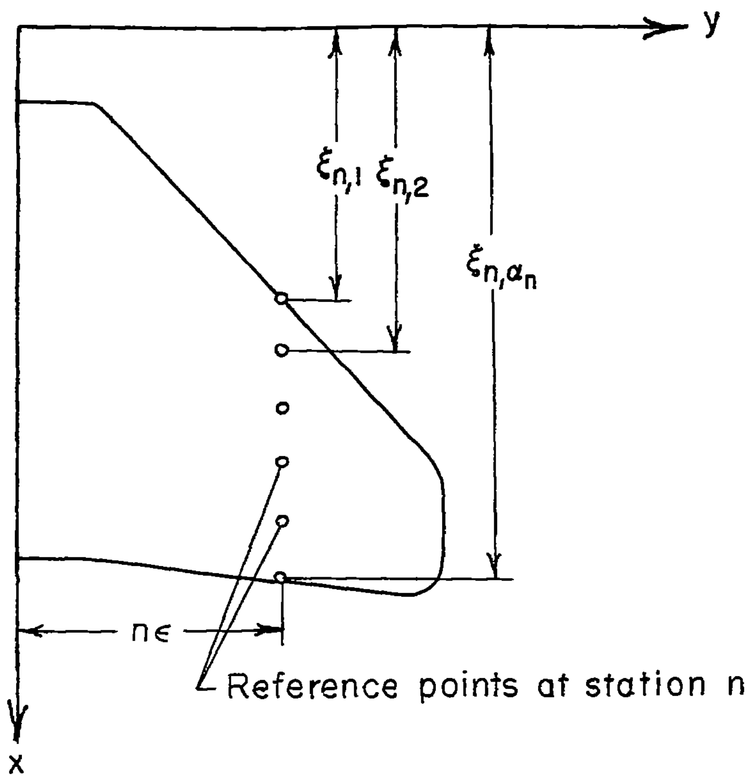

Figure 7.- Location of typical reference points on plan form of half-wing.